MTH4100 Calculus I

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L'Hôpital's Rule and Indeterminate Forms

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Theorem (L'Hôpital's Rule - Weak Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ both exist, and that $g'(a) \neq 0$. Then

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Example: Determine $\lim_{x \to 0} \dfrac{5x - \sin x}{x}$

L'Hôpital's Rule - Warning

Always check $f(a) = g(a) = 0$, before you try to use l'Hôpital to calculate $\lim_{x \to a} \frac{f(x)}{g(x)}$. Otherwise you may get a wrong answer.

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Example: $\displaystyle\lim_{x \to 0} \frac{1 + \sin x}{1 - x}$.

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Example: Determine

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}.$$

One-sided limits

L'Hôpital's rule can also be applied to one-sided limits.

Example:

$$\lim_{x \to 0^+} \frac{\sin x}{x^2}$$

Other indeterminate forms

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Example: Determine $\lim_{x\to\infty} \frac{x-x^2}{x^2+7x}$.

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$\infty \cdot 0$: If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x)$, then use $\lim_{x \to a}(f(x)g(x)) = \lim_{x \to a} \frac{g(x)}{1/f(x)}$.

Example: Determine $\lim_{x \to \infty} x \sin(1/x)$.

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$\infty - \infty$: Try to gather terms so we can use the standard form of L'Hôpital rule:
Example: Determine $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

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Antiderivatives

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Examples: (1) $f(x) = 2x$ (2) $h(x) = \sin x$.

It is easy to see that if $F(x)$ is an antiderivative of $f(x)$ then $F(x) + C$ will be an antiderivative of $f(x)$ for any constant $C \in \mathbb{R}$.

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Furthermore, if $G(x)$ is any other antiderivative of $f(x)$ then we have $F'(x) = f(x) = G'(x)$ and the second corollary to the Mean Value Theorem tells us that $G(x) = F(x) + C$ for some constant $C \in \mathbb{R}$. This gives:

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> If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is
>
> $$F(x) + C$$
>
> where C is an arbitrary constant.

Some common antideratives

TABLE 4.2 Antiderivative formulas

	Function	General antiderivative
1.	x^n	$\dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1, n$ rational
2.	$\sin kx$	$-\dfrac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\dfrac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

These formula can easily be verified by showing that the derivative of each antiderivative is equal to the given function.

Antiderivative linearity rules

Lemma

Suppose $f(x), g(x)$ are functions with antiderivatives $F(x)$ and $G(x)$, and $k \in \mathbb{R}$. Then:

- *$kf(x)$ has general antiderivative $kF(x) + C$;*
- *$f(x) + g(x)$ has general antiderivative $F(x) + G(x) + C$;*

for an arbitrary constant $C \in \mathbb{R}$.

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for an arbitrary constant $C \in \mathbb{R}$.

Example: Find the general antiderivative of $h(x) = \dfrac{5}{\sqrt{x}} + \sin 3x$.

Notation for antiderivatives

A special symbol is used to denote the collection of all antiderivatives of f:

DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\, dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

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Examples:

$\int 4x\,dx = 2x^2 + C$

$\int \cos x\,dx = \sin x + C$