# ! ${ }^{+}$Queen Mary University of London 

MTH4101 Calculus II<br>Lecture notes for Week 9 Integration III to IV

Thomas' Calculus, Sections 15.1 to 15.3

Rainer Klages<br>School of Mathematical Sciences<br>Queen Mary, University of London

Spring 2013

Consider the calculation of the volume under the plane $z=4-x-y$ over the rectangular region $R: 0 \leq x \leq 2$ and $0 \leq y \leq 1$ in the $x-y$ plane.
First consider a slice perpendicular to the $x$-axis:


The volume under the plane is

$$
\int_{x=0}^{x=2} A(x) \mathrm{d} x
$$

where $A(x)$ is the cross-sectional area at $x$. For each value of $x$ we may calculate $A(x)$ as the integral

$$
A(x)=\int_{y=0}^{y=1}(4-x-y) \mathrm{d} y
$$

which is the area under the curve $z=4-x-y$ in the plane of the cross-section at $x$. In calculating $A(x), x$ is held fixed and the integration takes place with respect to $y$. Combining the above two equations we have

$$
\begin{aligned}
\text { Volume } & =\int_{x=0}^{x=2} A(x) \mathrm{d} x \\
& =\int_{x=0}^{x=2}\left(\int_{y=0}^{y=1}(4-x-y) \mathrm{d} y\right) \mathrm{d} x \\
& =\int_{x=0}^{x=2}\left[4 y-x y-\frac{y^{2}}{2}\right]_{y=0}^{y=1} \mathrm{~d} x=\int_{x=0}^{x=2}\left(\frac{7}{2}-x\right) \mathrm{d} x \\
& =\left[\frac{7}{2} x-\frac{x^{2}}{2}\right]_{0}^{2}=\left(\frac{7}{2} \cdot 2-\frac{2^{2}}{2}\right)-(0-0)=5
\end{aligned}
$$

We can write

$$
\text { Volume }=\int_{0}^{2} \int_{0}^{1}(4-x-y) \mathrm{d} y \mathrm{~d} x
$$

This is an iterated or repeated integral. The expression states that we can get the volume under the plane by (i) integrating $4-x-y$ with respect to $y$ from $y=0$ to $y=1$, holding $x$ fixed, and then (ii) integrating the resulting expression in $x$ from $x=0$ to $x=2$. In other words, first do the $\mathrm{d} y$ integral and then do the $\mathrm{d} x$ integral.
Now consider the plane perpendicular to the $y$-axis:


We have

$$
A(y)=\int_{x=0}^{x=2}(4-x-y) \mathrm{d} x=\left[4 x-\frac{x^{2}}{2}-x y\right]_{x=0}^{x=2}=6-2 y .
$$

The volume is then

$$
\text { Volume }=\int_{y=0}^{y=1} A(y) \mathrm{d} y=\int_{y=0}^{y=1}(6-2 y) \mathrm{d} y=\left[6 y-y^{2}\right]_{0}^{1}=5
$$

as before.
This illustrates

## THEOREM 1 Fubini's Theorem (First Form)

If $f(x, y)$ is continuous throughout the rectangular region $R: a \leq x \leq b$, $c \leq y \leq d$, then

$$
\iint_{R} f(x, y) d A=\int_{c}^{d} \int_{a}^{b} f(x, y) d x d y=\int_{a}^{b} \int_{c}^{d} f(x, y) d y d x
$$

## Example:

Calculate the volume $V$ under $z=f(x, y)=x^{2} y$ over the rectangle $R$ defined by $1 \leq x \leq 2$, $-3 \leq y \leq 4$.

$$
\begin{aligned}
V & =\int_{R} \int x^{2} y \mathrm{~d} A=\int_{x=1}^{x=2}\left(\int_{y=-3}^{y=4} x^{2} y \mathrm{~d} y\right) \mathrm{d} x \\
& =\int_{x=1}^{x=2}\left[\frac{x^{2} y^{2}}{2}\right]_{y=-3}^{y=4} \mathrm{~d} x=\int_{x=1}^{x=2} \frac{7 x^{2}}{2} \mathrm{~d} x=\left[\frac{7 x^{3}}{6}\right]_{x=1}^{x=2}=\frac{49}{6} .
\end{aligned}
$$

Changing the order gives the same result:

$$
\begin{aligned}
V & =\int_{R} \int x^{2} y \mathrm{~d} A=\int_{y=-3}^{y=4}\left(\int_{x=1}^{x=2} x^{2} y \mathrm{~d} x\right) \mathrm{d} y \\
& =\int_{y=-3}^{y=4}\left[\frac{x^{3} y}{3}\right]_{x=1}^{x=2} \mathrm{~d} y=\int_{y=-3}^{y=4} \frac{7 y}{3} \mathrm{~d} y=\left[\frac{7 y^{2}}{6}\right]_{y=-3}^{y=4}=\frac{49}{6} .
\end{aligned}
$$

In this example we could have separated the integrand into its $x$ and $y$ parts:

$$
V=\int_{x=1}^{x=2}\left(\int_{y=-3}^{y=4} x^{2} y \mathrm{~d} y\right) \mathrm{d} x=\left(\int_{x=1}^{x=2} x^{2} \mathrm{~d} x\right)\left(\int_{y=-3}^{y=4} y \mathrm{~d} y\right)=\frac{7}{3} \cdot \frac{7}{2}=\frac{49}{6} .
$$

More generally, if $f(x, y)=g(x) h(y)$, (i.e. the function is separable) and the region is rectangular then

$$
\begin{aligned}
\int_{R} \int g(x) h(y) \mathrm{d} A & =\int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} g(x) h(y) \mathrm{d} y\right) \mathrm{d} x \\
& =\left(\int_{x=a}^{x=b} g(x) \mathrm{d} x\right)\left(\int_{y=c}^{y=d} h(y) \mathrm{d} y\right) .
\end{aligned}
$$

Now consider the case where the region $R$ is not rectangular: ${ }^{1}$

## THEOREM 2 Fubini's Theorem (Stronger Form)

Let $f(x, y)$ be continuous on a region $R$.

1. If $R$ is defined by $a \leq x \leq b, g_{1}(x) \leq y \leq g_{2}(x)$, with $g_{1}$ and $g_{2}$ continuous on $[a, b]$, then

$$
\iint_{R} f(x, y) d A=\int_{a}^{b} \int_{g_{1}(x)}^{g_{2}(x)} f(x, y) d y d x
$$

2. If $R$ is defined by $c \leq y \leq d, h_{1}(y) \leq x \leq h_{2}(y)$, with $h_{1}$ and $h_{2}$ continuous on $[c, d]$, then

$$
\iint_{R} f(x, y) d A=\int_{c}^{d} \int_{h_{1}(y)}^{h_{2}(y)} f(x, y) d x d y
$$

[^0]
## Example:

Find the volume of the prism $\iint_{R}(3-x-y) \mathrm{d} A$ where $R$ is the region bounded by the $x$-axis and the lines $x=1$ and $y=x$.


The region of integration in the $x-y$ plane and the volume defined by $z=3-x-y$ are shown in the figure. In order to do the double integral we will first consider the approach where we fix the value of $x$ and do the $y$ integral. We have

$$
\begin{aligned}
\int_{R} \int(3-x-y) \mathrm{d} A & =\int_{x=0}^{x=1} \int_{y=0}^{y=x}(3-x-y) \mathrm{d} y \mathrm{~d} x=\int_{0}^{1}\left[3 y-x y-\frac{y^{2}}{2}\right]_{y=0}^{y=x} \mathrm{~d} x \\
& =\int_{0}^{1}\left(3 x-\frac{3 x^{2}}{2}\right) \mathrm{d} x=\left[\frac{3 x^{2}}{2}-\frac{x^{3}}{2}\right]_{0}^{1}=1
\end{aligned}
$$

We can also change the order of the integration where we fix the value of $y$ and do the $x$ integral. We have

$$
\begin{aligned}
\int_{R} \int(3-x-y) \mathrm{d} A & =\int_{y=0}^{y=1} \int_{x=y}^{x=1}(3-x-y) \mathrm{d} x \mathrm{~d} y=\int_{0}^{1}\left[3 x-\frac{x^{2}}{2}-x y\right]_{x=y}^{x=1} \mathrm{~d} y \\
& =\int_{0}^{1}\left(\left(3-\frac{1}{2}-y\right)-\left(3 y-\frac{y^{2}}{2}-y^{2}\right)\right) \mathrm{d} y \\
& =\int_{0}^{1}\left(\frac{5}{2}-4 y+\frac{3}{2} y^{2}\right) \mathrm{d} y=\left[\frac{5}{2} y-2 y^{2}+\frac{y^{3}}{2}\right]_{y=0}^{y=1}=1
\end{aligned}
$$

In some cases the order of integration can be crucial to solving the problem.

## Example:

Calculate $\iint_{R}(\sin x) / x \mathrm{~d} A$ where $R$ is the triangle in the $x-y$ plane bounded by the $x$-axis, the line $y=x$ and the line $x=1$.


Taking vertical strips (i.e. keeping $x$ fixed and allowing $y$ to vary) gives

$$
\begin{aligned}
\int_{0}^{1}\left(\int_{0}^{x} \frac{\sin x}{x} \mathrm{~d} y\right) \mathrm{d} x & =\int_{0}^{1}\left[y \frac{\sin x}{x}\right]_{y=0}^{y=x} \mathrm{~d} x=\int_{0}^{1} \sin x \mathrm{~d} x \\
& =[-\cos x]_{0}^{1}=-\cos 1+\cos 0=1-\cos 1
\end{aligned}
$$

However, if we reverse the order of integration we get

$$
\int_{0}^{1} \int_{y}^{1} \frac{\sin x}{x} \mathrm{~d} x \mathrm{~d} y
$$

and $\int(\sin x) / x \mathrm{~d} x$ cannot be expressed in terms of elementary functions making the integral difficult to do.
There are always two ways to do a double integral; choose the simpler because the other may be impossible!
A key part of the process of double (and multiple) integration over a region is to find the limits of the integration. We can illustrate the procedure by considering the double integral of a function over the region $R$ given by the intersection of the line $x+y=1$ with the circle $x^{2}+y^{2}=1$.

1. Sketch the region of integration and label its boundary curves.
2. If we decide to use vertical cross-sections first: Find the $y$-limits of integration. Imagine a vertical line through the region, $R$, and mark the points where it enters and leaves $R$. In this case such a line would enter at $y=1-x$ and leave at $y=\sqrt{1-x^{2}}$.
3. Find the $x$-limits of integration: Choose the $x$-limits that include all vertical lines through $R$. In this case the lower limit is $x=0$ and the upper limit is $x=1$.
4. This step may not be necessary: Reversing the order of integration. Then the $x$-limits would be from $x=1-y$ to $x=\sqrt{1-y^{2}}$ and the $y$-limits from $y=0$ to $y=1$.


## Example:

Sketch the region of integration for the integral

$$
\int_{0}^{2} \int_{x^{2}}^{2 x}(4 x+2) \mathrm{d} y \mathrm{~d} x
$$

and write an equivalent integral with the order of integration reversed. Evaluate the integral.

(a)

(b)

As written, the order of integration would imply that we do the $y$-integral first, from $y=x^{2}$ to $y=2 x$, followed by the $x$-integral from $x=0$ to $x=2$. However, we are told to reverse
the order of integration. This means we do the $x$-integration first, from $x=y / 2$ to $x=\sqrt{y}$, followed by the $y$-integral from $y=0$ to $y=4$. In other words,

$$
\int_{0}^{2} \int_{x^{2}}^{2 x}(4 x+2) \mathrm{d} y \mathrm{~d} x=\int_{0}^{4} \int_{y / 2}^{\sqrt{y}}(4 x+2) \mathrm{d} x \mathrm{~d} y
$$

We can evaluate the integral using either ordering. Let us revert to the original:

$$
\begin{aligned}
\int_{0}^{2} \int_{x^{2}}^{2 x}(4 x+2) \mathrm{d} y \mathrm{~d} x & =\int_{0}^{2}[4 x y+2 y]_{x^{2}}^{2 x} \mathrm{~d} x=\int_{0}^{2}\left(8 x^{2}+4 x-4 x^{3}-2 x^{2}\right) \mathrm{d} x \\
& =\int_{0}^{2}\left(-4 x^{3}+6 x^{2}+4 x\right) \mathrm{d} x=\left[-x^{4}+2 x^{3}+2 x^{2}\right]_{0}^{2} \\
& =-16+16+8=8
\end{aligned}
$$

Note that this example is not separable because it is a non-rectangular region (i.e. the limits on the $x$ and $y$ integrals now depend on the region of integration).

Double integrals can also be calculated over unbounded regions.

## Example:

Evaluate the integral $\int_{0}^{\infty} \int_{0}^{\infty} x e^{-(x+2 y)} \mathrm{d} x \mathrm{~d} y$.
We have

$$
\begin{aligned}
\int_{0}^{\infty} \int_{0}^{\infty} x e^{-(x+2 y)} \mathrm{d} x \mathrm{~d} y & =\int_{0}^{\infty} \int_{0}^{\infty} e^{-2 y} x e^{-x} \mathrm{~d} x \mathrm{~d} y \\
& =\int_{0}^{\infty} e^{-2 y}\left\{\left[-x e^{-x}\right]_{0}^{\infty}-\int_{0}^{\infty}\left(-e^{-x}\right) \mathrm{d} x\right\} \mathrm{d} y \\
& =\int_{0}^{\infty} e^{-2 y}\left((0-0)+\left[-e^{-x}\right]_{0}^{\infty}\right) \mathrm{d} y \\
& =\left[-\frac{1}{2} e^{-2 y}\right]_{0}^{\infty}=0-\left(-\frac{1}{2}\right)=\frac{1}{2} .
\end{aligned}
$$

Double integrals have the following properties:
Let $f(x, y), g(x, y)$ be continuous on the bounded region $R$. Then

$$
\begin{aligned}
& \int_{R} \int c f(x, y) \mathrm{d} A=c \int_{R} \int f(x, y) \mathrm{d} A \quad \text { for any number } c \\
& \int_{R} \int(f(x, y) \pm g(x, y)) \mathrm{d} A=\int_{R} \int f(x, y) \mathrm{d} A \pm \int_{R} \int g(x, y) \mathrm{d} A \\
& \int_{R} \int f(x, y) \mathrm{d} A \geq 0 \quad \text { if } \quad f(x, y) \geq 0 \text { on } R \\
& \int_{R} \int f(x, y) \mathrm{d} A \geq \int_{R} \int g(x, y) \mathrm{d} A \text { if } f(x, y) \geq g(x, y) \text { on } R \\
& \int_{R} \int f(x, y) \mathrm{d} A=\int_{R_{1}} \int f(x, y) \mathrm{d} A+\int_{R_{2}} \int f(x, y) \mathrm{d} A \\
& \text { if } R=R_{1} \cup R_{2}, R_{1} \cap R_{2}=\emptyset
\end{aligned}
$$

## Area by double integration

The area $A$ of a closed, bounded plane region $R$ is given by

$$
A=\lim _{n \rightarrow \infty} \sum_{k=1}^{n} \Delta A_{k}=\int_{R} \int \mathrm{~d} A,
$$

which is equivalent to calculating $\iint_{R} f(x, y) \mathrm{d} A$ with $f(x, y)=1$.


## Example:

Find the area of the region $R$ enclosed by the parabola $y=x^{2}$ and the line $y=x+2$.
Determining the points of intersection is essential to determining the limits on the integrations. We can find the points by setting $x^{2}=x+2$ which gives $x^{2}-x-2=(x+1)(x-2)=0$, giving $x=-1$ and $x=2$. The corresponding values of $y$ are $y=1$ and $y=4$. So the points of intersection are $(-1,1)$ and $(2,4)$.

(a)

(b)

If we use vertical strips (i.e. fix $x$ and vary $y$ ) for the first integral we will not have to split up the region of integration. From the diagram we see that the lower and upper limits for the first integration are therefore $y=x^{2}$ and $y=x+2$. This gives

$$
\begin{aligned}
A & =\int_{-1}^{2} \int_{x^{2}}^{x+2} \mathrm{~d} y \mathrm{~d} x=\int_{-1}^{2}[y]_{x^{2}}^{x+2} \mathrm{~d} x \\
& =\int_{-1}^{2}\left(x+2-x^{2}\right) \mathrm{d} x=\left[\frac{x^{2}}{2}+2 x-\frac{x^{3}}{3}\right]_{-1}^{2}=\frac{9}{2} .
\end{aligned}
$$

Double integrals can also be used to find the average value of the function $f(x, y)$ over the region $R$, which is defined to be

$$
\langle f\rangle=\frac{1}{\text { area of } R} \int_{R} \int f(x, y) \mathrm{d} A
$$

## Example:

Find the average value of $f(x, y)=x \cos x y$ over the rectangle $R: 0 \leq x \leq \pi, 0 \leq y \leq 1$.
The area of the region $R$ is just $\pi$, the product of the length of the two sides of the rectangle.
We just need to find $\iint_{R} f(x, y) \mathrm{d} A$ and then divide by $\pi$.

$$
\begin{aligned}
\int_{0}^{\pi} \int_{0}^{1} x \cos x y \mathrm{~d} y \mathrm{~d} x & =\int_{0}^{\pi}[\sin x y]_{y=0}^{y=1} \mathrm{~d} x \\
& =\int_{0}^{\pi}(\sin x-0) \mathrm{d} x=[-\cos x]_{0}^{\pi}=1+1=2
\end{aligned}
$$

Hence $\langle f\rangle=2 / \pi$.


[^0]:    ${ }^{1}$ see Thomas' Calculus, beginning of Section 15.2 for details underlying this theorem

