MTH4101 Calculus II

Lecture notes for Week 4
Derivatives, V, Series I

Thomas' Calculus, Sections 14.8, 10.1 and 10.2

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Lagrange Multipliers

We now consider the problem to find extrema of a function $f(x, y, z)$ whose domain is constrained by another function $g(x, y, z) = 0$ to lie within some subset.

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when $g(x, y, z) = 0$. To find the **local maximum and minimum values** of f **subject to the constraint** $g(x, y, z) = 0$, we need to find the values of x, y, z and λ that simultaneously satisfy the equations

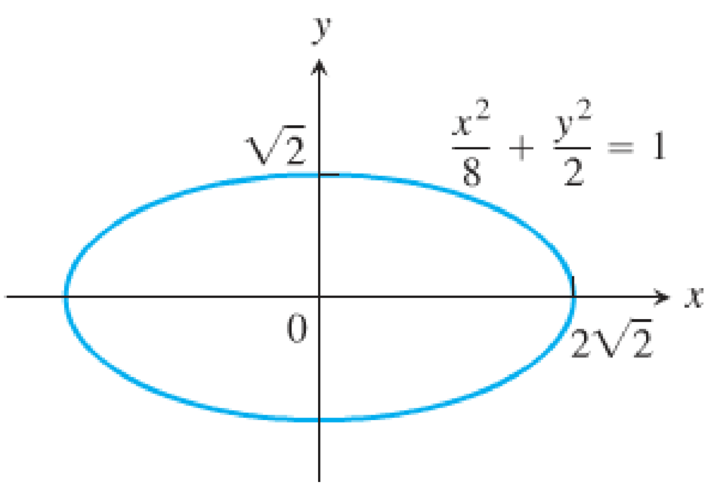
$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 0\,.$$

This is the **Method of Lagrange Multipliers**. For functions of two variables the condition is similar but without the variable z.

We will see how the method works by considering two examples.[1]

Example:
Find the greatest and smallest values that the function $f(x, y) = xy$ takes on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1\,.$$



We need to find the extreme values of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0\,.$$

First, find the values of x, y and λ for which

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y) = 0\,.$$

[1] A detailed motivation and a sketch of the proof are provided in Thomas' Calculus, beginning of Section 14.8.

$$\begin{aligned}\nabla f &= f_x\mathbf{i} + f_y\mathbf{j} = y\mathbf{i} + x\mathbf{j}\\ \nabla g &= g_x\mathbf{i} + g_y\mathbf{j} = \frac{x}{4}\mathbf{i} + y\mathbf{j}\,.\end{aligned}$$

Hence

$$y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4}x\mathbf{i} + \lambda y\mathbf{j}\,.$$

Comparing components gives

$$y = \frac{\lambda}{4}x\,, \qquad x = \lambda y\,.$$

Therefore

$$y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4}y\,.$$

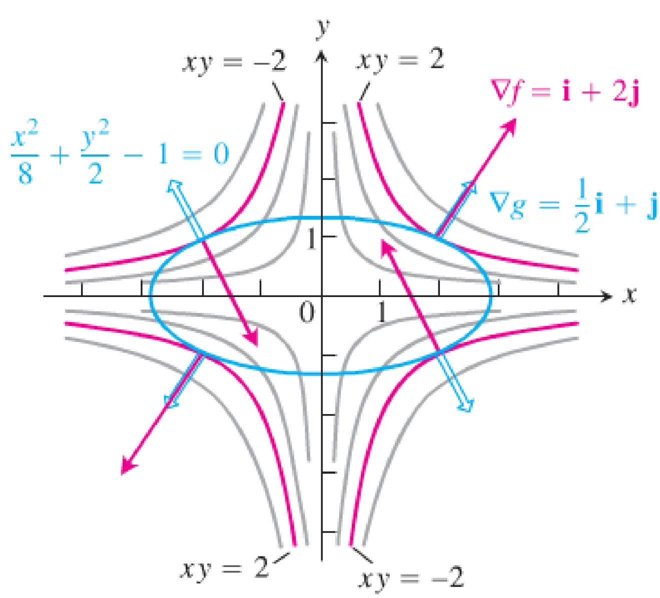
Hence $y = 0$ or $\lambda = \pm 2$ and there are two cases to consider.

1. If $y = 0$, then $x = y = 0$. But $(0,0)$ does not lie on the ellipse, hence $y \neq 0$.

2. If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting in $g(x,y) = 0$ gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1 \qquad \Rightarrow 4y^2 + 4y^2 = 8 \qquad \Rightarrow y = \pm 1\,.$$

Therefore $f(x,y)$ has its extreme values on the ellipse at the four points $(\pm 2, 1)$, $(\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.
The level curves of $f(x,y) = xy$ are the hyperbolas $xy = c$. The extreme values are the points on the ellipse when ∇f (red) is a scalar multiple of ∇g (blue):



Example:
Find the maximum and minimum values of the function $f(x,y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

$$f(x,y) = 3x + 4y\,, \qquad g(x,y) = x^2 + y^2 - 1\,.$$

The Lagrange multiplier condition states that $\nabla f = \lambda \nabla g$, hence

$$3\mathbf{i} + 4\mathbf{j} = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j} \qquad \Rightarrow x = \frac{3}{2\lambda}\,, \quad y = \frac{2}{\lambda} \qquad (\lambda \neq 0\,;\ \text{why?})\,.$$

Therefore x and y have the same sign.
The condition $g(x,y) = 0$ gives

$$x^2 + y^2 - 1 = 0$$

and this gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0\,.$$

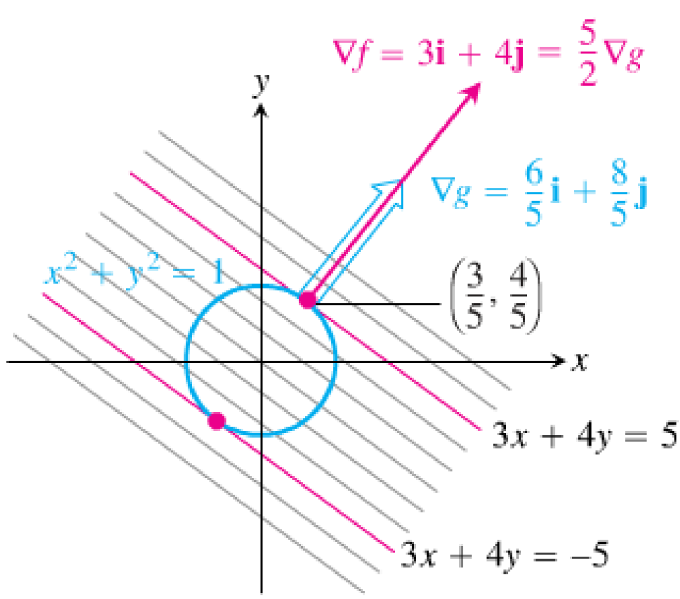
This gives

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1 \qquad \Rightarrow 9 + 16 = 4\lambda^2 \qquad \Rightarrow \lambda = \pm\frac{5}{2}\,.$$

Hence

$$x = \frac{3}{2\lambda} = \pm\frac{3}{5}\,, \qquad y = \frac{2}{\lambda} = \pm\frac{4}{5}\,.$$

Therefore the function $f(x,y) = 3x + 4y$ has extreme values at $(x,y) = \pm(3/5, 4/5)$.
The level curves of $f(x,y) = 3x + 4y$ are the lines $3x + 4y = c$. The further the lines lie from the origin, the larger the absolute value of f:



Sequences

A **sequence** is a list of numbers in a given *order*:

$$a_1, a_2, a_3, \ldots, a_n, \ldots .$$

Each of the a_1, a_2, etc. represents a number; these are the *terms* of the sequence. For example

$$2, 4, 6, 8, \ldots, 2n, \ldots$$

has first term $a_1 = 2$, second term $a_2 = 4$ and nth term $a_n = 2n$. The integer n is called the *index* of a_n and denotes where a_n occurs in the list.
We can consider the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ as a function that sends 1 to a_1, 2 to a_2, etc. and in general sends the positive integer n to the nth term a_n.

DEFINITION Infinite Sequence
An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

Sequences can be described by *rules* or by *listing terms*. For example,

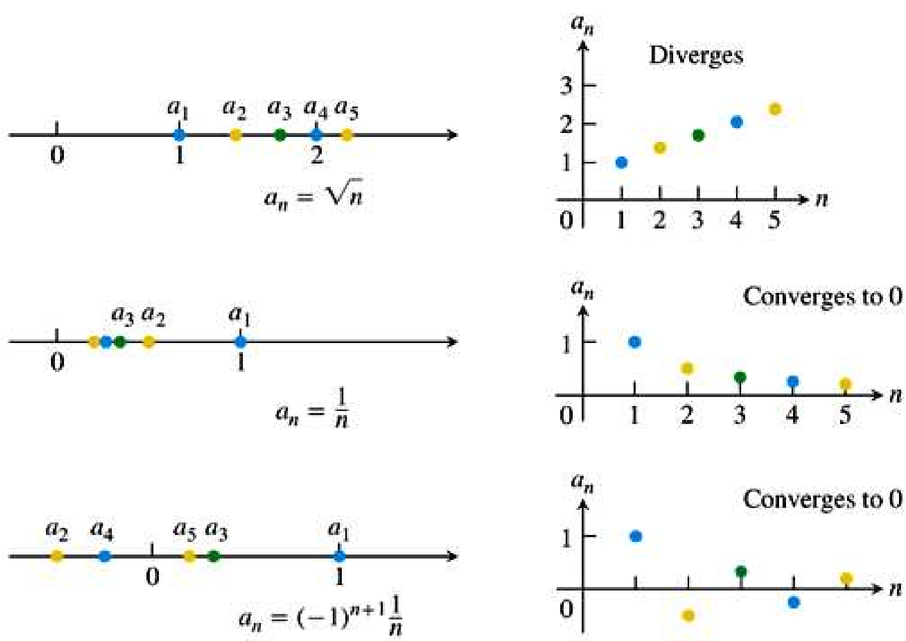
$$a_n = \sqrt{n} \qquad \{a_n\} = \left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\right\}$$

$$b_n = (-1)^{n+1}(1/n) \qquad \{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\right\}$$

$$c_n = (n-1)/n \qquad \{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots\right\}$$

$$d_n = (-1)^{n+1} \qquad \{d_n\} = \left\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\right\}$$

Sequences can be illustrated graphically either as points on a real axis or as the graph of a function defining the sequence:



Consider the following sequences:

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\} \qquad \text{terms approach 0 as } n \text{ gets large}$$

$$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1-\frac{1}{n}, \ldots\right\} \qquad \text{terms approach 1 as } n \text{ gets large}$$

$$\left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots, \sqrt{n}, \ldots\right\} \qquad \text{terms get larger than any number as } n \text{ increases}$$

$$\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\} \qquad \text{terms oscillate between 1 and } -1,$$

never converging to a single value

This leads to the definition of **convergence**, **divergence** and a **limit**:

DEFINITIONS Converges, Diverges, Limit

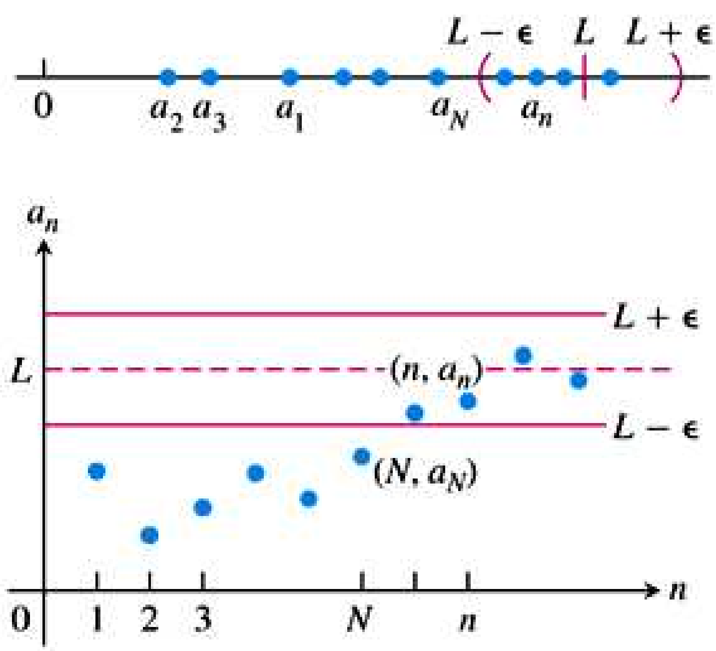
The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \qquad \Rightarrow \qquad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence

The concept of a limit is illustrated in the following figure:



Here $a_n \to L$ if $y = L$ is a horizontal asymptote of the sequence of points $\{(n, a_n)\}$.

We will now consider two examples of the application of the definitions.

Example:
We want to prove that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Let $\epsilon > 0$ be given. We need to find an integer N such that for all n,

$$n > N \quad \Rightarrow \quad \left|\frac{1}{n} - 0\right| < \epsilon\,.$$

This condition will be satisfied provided $1/n < \epsilon$, which means $n > 1/\epsilon$. Therefore if N is any integer greater than $1/\epsilon$, the implication will hold for all $n > N$. Hence $\lim_{n\to\infty}(1/n) = 0$. For example, suppose we take $\epsilon = 0.01$ then the condition is just $n > 100$.

Example:
We want to prove that the sequence

$$\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\} \quad \text{diverges.}$$

proof by contradiction: Assume that the sequence converges to some number L. Choose $\epsilon = \frac{1}{2}$ in the definition of the limit and so all terms a_n of the sequence with n larger than some N must lie within $\epsilon = \frac{1}{2}$ of L:

$$n > N \quad \Rightarrow \quad |a_n - L| < \frac{1}{2}\,.$$

Since 1 is in every other term of the sequence, 1 must lie within ϵ of L. Hence

$$|1 - L| = |L - 1| < \frac{1}{2} \quad \text{or} \quad \frac{1}{2} < L < \frac{3}{2}\,.$$

Then -1 is also in every other term and so we must have

$$|L - (-1)| < \frac{1}{2} \quad \text{or} \quad -\frac{3}{2} < L < -\frac{1}{2}\,.$$

However, this is a *contradiction*: Both conditions cannot be satisfied simultaneously. Therefore no such limit exists and so the sequence diverges.
There is a second type of divergence:

DEFINITION Diverges to Infinity
The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad a_n \to \infty\,.$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \qquad \text{or} \qquad a_n \to -\infty\,.$$

Example:

$$\lim_{n\to\infty} \sqrt{n} = \infty \quad \text{(proof?)}$$

note: The sequence $\{1, -2, 3, -4, 5, \ldots\}$ *also* diverges, but not to ∞ or $-\infty$.

Sequences are functions with domain restricted to $n \in \mathbb{N}$, hence:

THEOREM 1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

We can use these rules to help us to calculate limits of sequences.

Example:
Find $\lim\limits_{n\to\infty} \dfrac{n-1}{n}$.

$$\lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n} = 1 - 0 = 1\,.$$

Example:
Find $\lim\limits_{n\to\infty} \dfrac{5}{n^2}$.

$$\lim_{n\to\infty} \frac{5}{n^2} = 5 \cdot \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 5 \cdot 0 \cdot 0 = 0\,.$$

The **Sandwich Theorem for Sequences** provides another method for finding the limits of sequences:

THEOREM 2 The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Note that if $|b_n| \le c_n$ and $c_n \to 0$ as $n \to \infty$, then $b_n \to 0$ also, because $-c_n \le b_n \le c_n$.

Example:
Find $\lim\limits_{n\to\infty} \dfrac{\sin n}{n}$.
By the properties of the sine function we have $-1 \le \sin n \le 1$ for all n. Therefore

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \qquad \Rightarrow \qquad \lim_{n\to\infty} \frac{\sin n}{n} = 0$$

because of $\lim_{n\to\infty}(-1/n) = \lim_{n\to\infty}(1/n) = 0$ and the use of the Sandwich Theorem.

Example:

Find $\lim\limits_{n\to\infty} \dfrac{1}{2^n}$.

$1/2^n$ must always lie between 0 and $1/n$ (e.g. $\frac{1}{2} < 1, \frac{1}{4} < \frac{1}{2}, \frac{1}{8} < \frac{1}{3}, \frac{1}{16} < \frac{1}{4}, \ldots$). Therefore

$$0 \leq \frac{1}{2^n} \leq \frac{1}{n} \qquad \Rightarrow \qquad \lim_{n\to\infty} \frac{1}{2^n} = 0\,.$$

The limits of sequences can also be determined by using the following theorem:

THEOREM 3 The Continuous Function Theorem for Sequences
Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Example:

Determine the limit of the sequence $\left\{2^{1/n}\right\}$ as $n \to \infty$.

We already know that the sequence $\left\{\frac{1}{n}\right\}$ converges to 0 as $n \to \infty$. Let $a_n = 1/n$, $f(x) = 2^x$ and $L = 0$ in the continuous function theorem for sequences. This gives

$$2^{1/n} = f(1/n) \to f(L) = 2^0 = 1 \quad \text{as} \quad n \to \infty\,.$$

Hence the sequence $\left\{2^{1/n}\right\}$ converges to 1.

We can also make use of l'Hôpital's Rule to find the limits of sequences. To do so we need to make use of the following theorem:

THEOREM 4
Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \qquad \Rightarrow \qquad \lim_{n\to\infty} a_n = L.$$

Example:

Show that $\lim\limits_{n\to\infty} \dfrac{\ln n}{\sqrt{n}} = 0$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} &= \lim_{n\to\infty} \frac{1/n}{(1/2)n^{-1/2}} \\
&\quad \text{(using l'Hôpital's Rule \textit{by treating n as a continuous real variable})} \\
&= \lim_{n\to\infty} 2 \cdot \frac{n^{1/2}}{n} = 2 \lim_{n\to\infty} \frac{1}{n^{1/2}} = 0\,.
\end{aligned}$$