MTH4100 Calculus I

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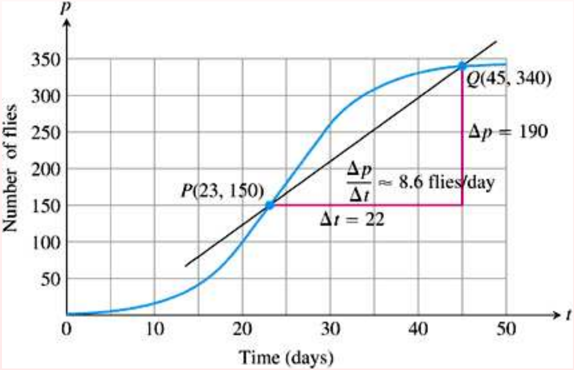
Example - average rate of change

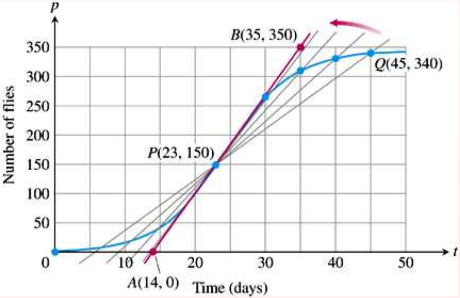
Growth of a fruit fly population measured experimentally. It is straightforward to calculate the average rate of change from day 23 to day 45.



Example - Instantaneous rate of change

We can also calculate the *instantaneous rate of change* at a particular time on a specific day, e.g. 00:00 on day 23, by finding the average rates of change over *increasingly short time intervals* starting at time 00:00 on day 23:

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\dfrac{340-150}{45-23} \approx 8.6$
(40, 330)	$\dfrac{330-150}{40-23} \approx 10.6$
(35, 310)	$\dfrac{310-150}{35-23} \approx 13.3$
(30, 265)	$\dfrac{265-150}{30-23} \approx 16.4$



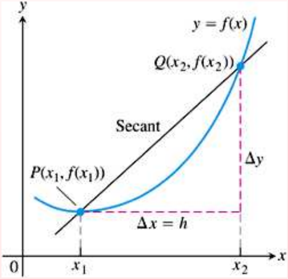
The lines PQ approach the red *tangent* AB at the point P with slope

$$\frac{350-0}{35-14} \simeq 16.7 \text{ flies/day}$$

Definition The *average rate of change* of a function f over an interval $I = [x_1, x_2]$ is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h} \tag{1}$$

where $h = x_2 - x_1 \neq 0$.



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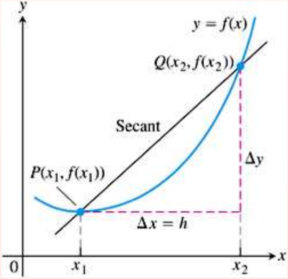

To move from average rates of change to instantaneous rates of change we need to consider 'the limiting value' of (1) as h approaches zero.

Informal Definition Let f be a function defined everywhere in an open interval containing x_0 (except possibly at x_0 itself). If $f(x)$ gets 'arbitrarily close to' a number L for all x 'sufficiently close to' but not equal to x_0, then we say that f *approaches the* limit L *as* x *approaches* x_0, and we write

$$\lim_{x \to x_0} f(x) = L\,.$$

This is read as "the limit of $f(x)$ as x approaches x_0 is equal to L."

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This definition is 'informal' because the terms "arbitrarily close to" and "sufficiently close to" are not precise. It will serve our purpose for this module (which is to get an intuitive understanding of limits). But it is still worthwhile to compare this informal definition with the precise definition.

Example

How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave as x approaches 1?
We can *simplify* this formula for $f(x)$ when $x \neq 1$. We have:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \text{ for } x \neq 1\,.$$

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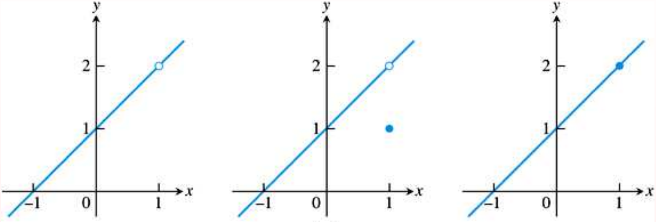
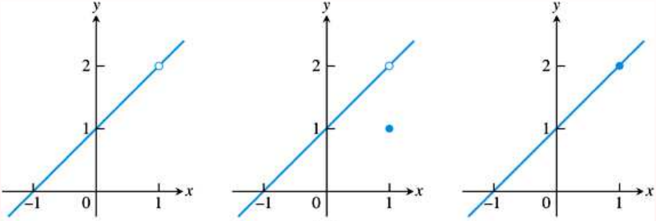
This *suggests* that

$$\lim_{x \to 1} f(x) = \lim_{x \to 1}(x+1) = 1 + 1 = 2.$$

Example continued



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$



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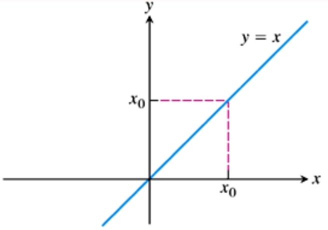
Note: The limit of a function at a point x_0 does not depend on the value the function takes at x_0. The function f in the above example is not defined at $x = 1$. But if we define a new function by choosing an arbitrary value for $f(1)$, then the new function will still have the same limit at $x = 1$ as f, see graphs (b) and (c) above. Note, however, that only the function h shown in (c) has the property that the limit value and the function value at $x = 1$ are the same i.e. $\lim_{x \to 1} h(x) = h(1)$.

Functions with limits everywhere

Some functions can have limits at all point on the real line:
(a) $f(x) = x$.

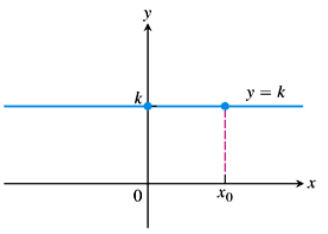


For any $x_0 \in \mathbb{R}$ we have $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$.

In particular

$$\lim_{x \to 3} x = 3 \,.$$

Some functions can have limits at all point on the real line:
(b) $f(x) = k$ for some constant $k \in \mathbb{R}$.



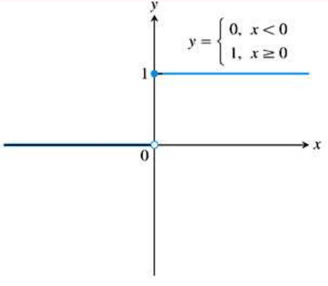
For any $x_0 \in \mathbb{R}$ we have $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k$.

For example, when $k = 5$, we have

$$\lim_{x \to -12} 5 = \lim_{x \to 7} 5 = 5\,.$$

The following functions have no limit at $x = 0$.

(a)



limit fails to exist because the function *jumps* as we approach $x = 0$.

Functions without a limit at some point

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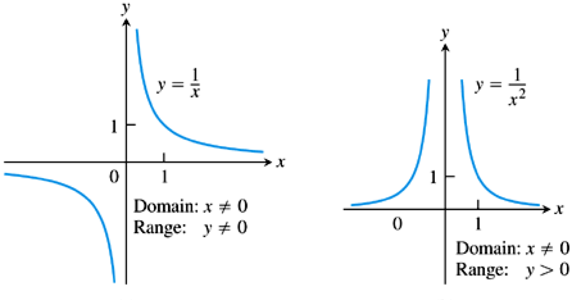
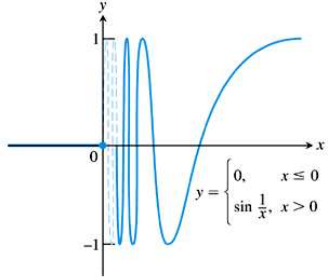


limit fails to exist because the functions *becomes too large* as we approach $x = 0$.

The following functions have no limit at $x = 0$.
(c)



limit fails to exist because the function *oscillates too much* as we approach $x = 0$ from the right.

Theorem

Suppose that c, L, M *are real numbers, and* f *and* g *are functions such that* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$. *Then:*

Limit laws

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1. Sum Rule: $\lim\limits_{x \to c} (f(x) + g(x)) = L + M$

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Suppose that c, L, M are real numbers, and f and g are functions such that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then:

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2. Difference Rule: $\lim\limits_{x\to c}(f(x)-g(x)) = L - M$
3. Constant Multiple Rule: $\lim\limits_{x\to c}(kf(x)) = kL$ *for any constant* $k \in \mathbb{R}$.

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6. Power Rule: $\lim\limits_{x\to c}(f(x)^{r/s}) = L^{r/s}$ *for any integers* r, s *such that* $L^{r/s}$ *is a real number.*

Limits of polynomial and rational functions

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THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

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THEOREM 3 **Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero**

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

‘Zero over zero’

Sometimes the numerator and denominator of a rational function can *both* become zero when we substitute a value of x. If this happens, we can try to first use an ‘algebraic simplification’ which gives us a non-zero denominator, and then calculate the limit by substitution.

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