MTH4101 Calculus II

Lecture notes for Week 1
Derivatives IV

Thomas' Calculus, Sections 14.1 to 14.2

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Spring 2013

Functions of Several Variables

Reminder: What is a function?
In *Calculus 1* and in *Mathematical Structures* you have learned the following:

Definition
A **function** from a set D (domain) to a set Y (codomain) is a rule that assigns a *unique* (single) element $y \in Y$ to each element $x \in D$.

So far you have dealt with functions of a *single* variable, such as

$$f : \mathbb{R} \to \mathbb{R} \quad , \quad x \mapsto y = f(x)$$

with, for example, $f(x) = x^2$.
Functions of *several variables* are defined in complete analogy to functions of one variable in terms of uniqueness, domain, codomain, range, etc. (without involving complex numbers):

DEFINITIONS Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

In the following we will focus on functions of two variables.

Examples:

$$\begin{aligned} V &= V(r,h) = \pi r^2 h \quad \text{(volume of cylinder, radius } r\text{, height } h) \\ M &= M(r,\rho) = \frac{4}{3}\pi r^3 \rho \quad \text{(mass of sphere, radius } r\text{, density } \rho) \end{aligned}$$

In the case of V the quantities r and h are the input (*independent*) variables and V is the *unique* output (*dependent*) variable.

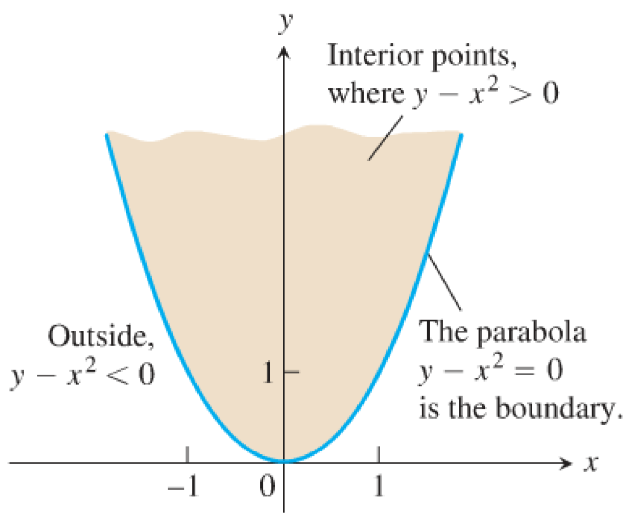
If f is a function of two independent variables, x and y, the domain of f is a region in the x-y plane.

Example:
(Natural) domains and ranges for function of two variables

Function	Domain	Range
$w = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$w = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$w = \sin xy$	Entire plane	$[-1, 1]$

Interior points, boundary points, open and closed sets are defined in higher dimensions in analogy to dealing with intervals on the real line.[1]

Example:
Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$.
Since f is defined only where $y - x^2 \geq 0$, the domain is the *closed* (the set contains all boundary points), *unbounded* (why?) region shown below (shaded). The parabola $y = x^2$ is the boundary of the domain. The points above the parabola make up the domain's interior.



There are two ways to visualise a function $f(x, y)$:

1. Sketch the **graph**, or **surface** $z = f(x, y)$ in space.

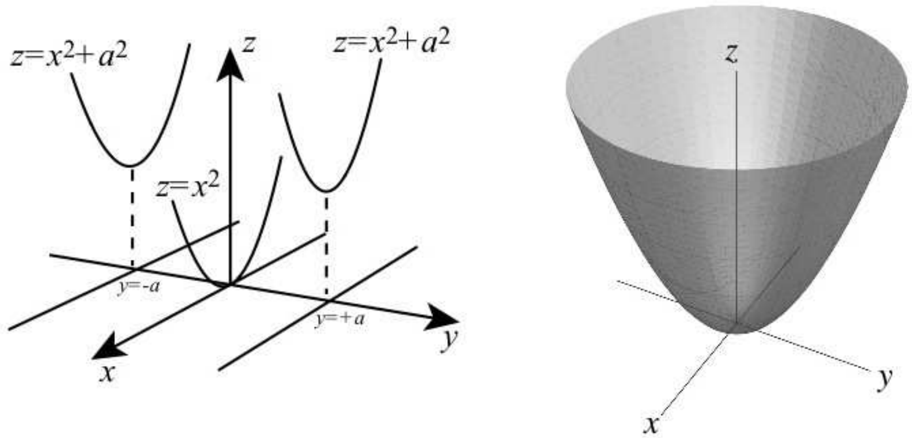
2. Draw and label **level curves** in the domain on which f has a constant value.

As an example for 1., we will consider the function

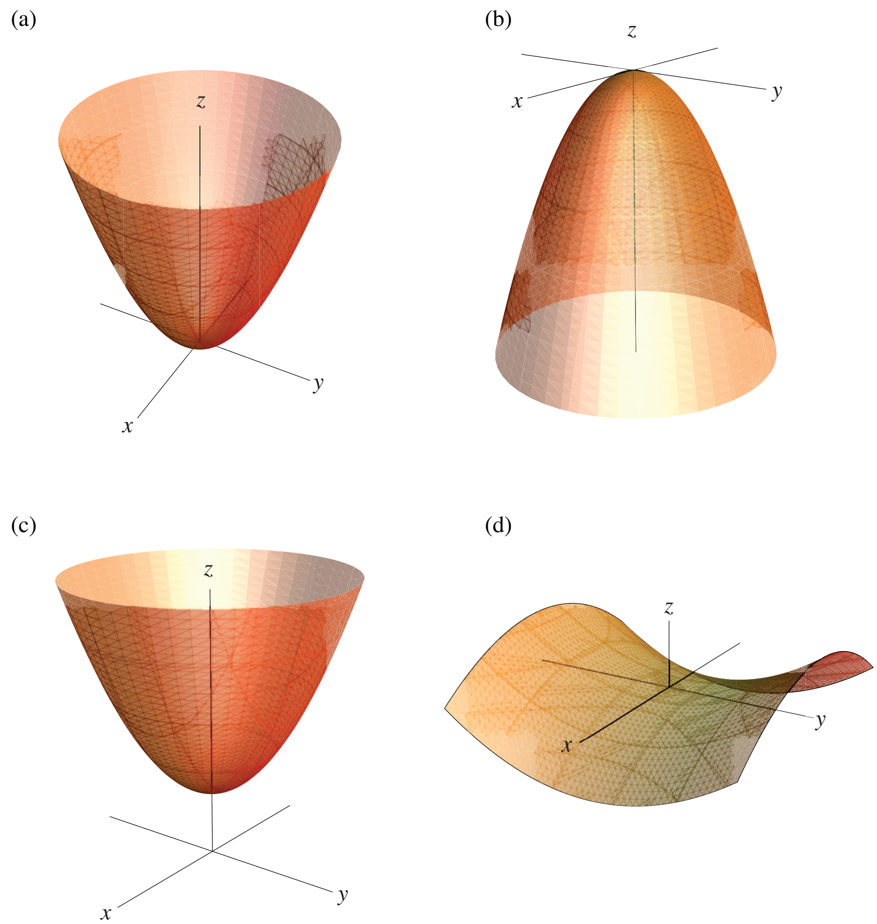
$$f(x, y) = x^2 + y^2 .$$

To visualise the surface, consider the nature of f for a fixed value of y, say $y = a$. In this case $z = x^2 + a^2$ and $z = z(x)$. The equation $z = x^2 + a^2$ defines a parabola in the plane $y = a$, perpendicular to the y-axis. Each different value of a gives a different parabola. For example, for $y = a = 0$ we have $z = x^2$. Therefore the required surface is made up of parabolas and forms a *paraboloid* as shown below.

[1] If you are not satisfied with this statement, please check out Thomas' Calculus p.749 for details.



Examples of other surfaces are shown in the following figure. It displays the three dimensional surfaces defined by the functions (a) $f(x, y) = x^2 + y^2$, (b) $f(x, y) = -x^2 - y^2$, (c) $f(x, y) = x^2 + y^2 + 5$ and (d) $f(x, y) = y^2 - x^2$.

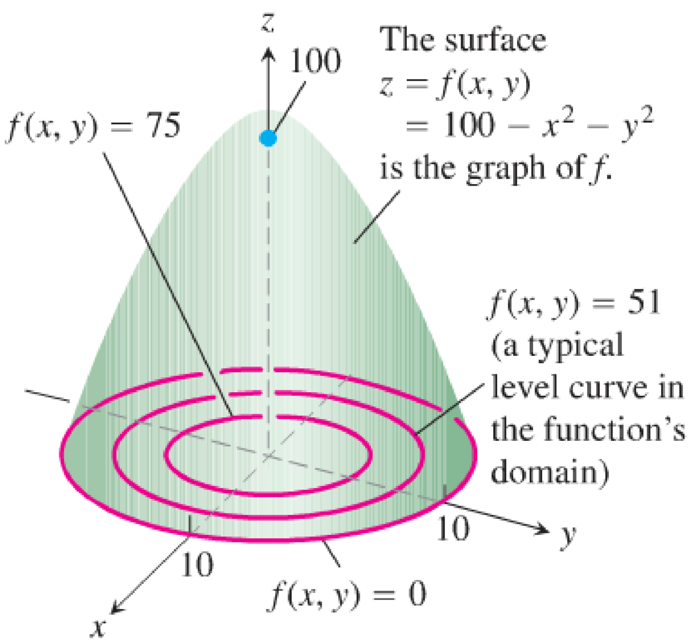


The set of points in the x-y plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f (cf. what is plotted in geographic maps).

Example:

Graph the function $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$ and $f(x, y) = 75$ in the domain of f in the plane.

The domain is the entire x-y plane and the range is the set of real numbers ≤ 100. The graph is the paraboloid given by $z = 100 - x^2 - y^2$:

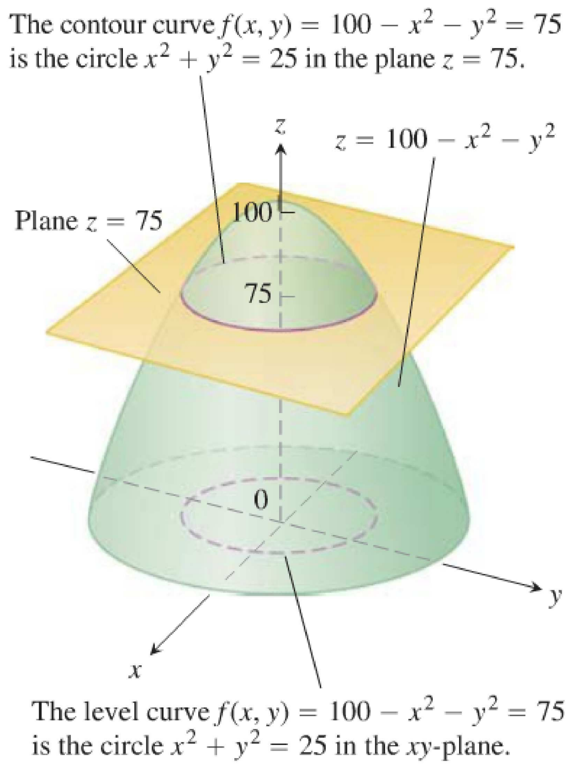


When $f(x, y) = 0$, we have $100 - x^2 - y^2 = 0$ or $x^2 + y^2 = 100$. This corresponds to a circle of radius 10.

When $f(x, y) = 51$, we have $100 - x^2 - y^2 = 51$ or $x^2 + y^2 = 49$. This corresponds to a circle of radius 7.

When $f(x, y) = 75$, we have $100 - x^2 - y^2 = 75$ or $x^2 + y^2 = 25$. This corresponds to a circle of radius 5.

The curve in space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is called the **contour curve** $f(x, y) = c$. The following figure shows the contour curve produced where the plane $z = 75$ intersects the surface $z = f(x, y) = 100 - x^2 - y^2$.



Limits and Continuity in Higher Dimensions

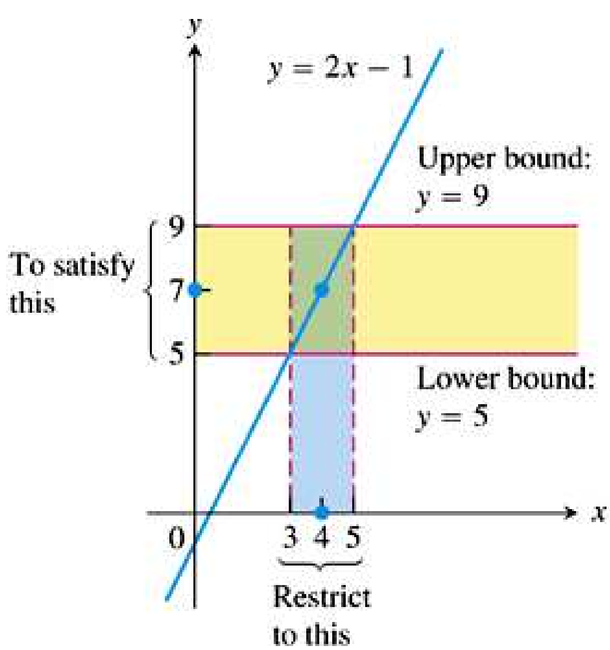
Reminder: Limits

For functions of one variable we say that $f(x)$ approaches the **limit** L whenever $f(x)$ is *arbitrarily close* to L for all x *sufficiently close* to a, written as

$$\lim_{x \to a} f(x) = L\,.$$

Example:

$\lim\limits_{x \to 4}(2x - 1) = 7.$



Analogously, if the values of $f(x,y)$ lie *arbitrarily close* to a fixed real number L for all points (x,y) *sufficiently close* to a point (x_0,y_0), we say that f approaches the limit L as (x,y) approaches (x_0,y_0). More rigorously:[2]

DEFINITION **Limit of a Function of Two Variables**

We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x,y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

It can be shown that this definition leads to the following properties (you have seen an analogous theorem for functions of one variable in Calculus 1):

Theorem *Properties of limits of functions of two variables*
If $L, M, k \in \mathbb{R}$, $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$ then

1. $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \pm g(x,y)) = L \pm M$
2. $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$
3. $\lim_{(x,y)\to(x_0,y_0)} (kf(x,y)) = kL$
4. $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$, $M \neq 0$
5. If r and s are integers with no common factors, and $s \neq 0$, then
   $\lim_{(x,y)\to(x_0,y_0)} (f(x,y))^{r/s} = L^{r/s}$ provided $L^{r/s}$ is a real number.

For polynomials and rational functions the limit as $(x,y) \to (x_0,y_0)$ can be calculated by evaluating the function at (x_0,y_0) (provided the rational function is defined at (x_0,y_0)).

Examples:
(1)

$$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3\,.$$

(2) Find

$$\lim_{(x,y)\to(0,0),x\neq y} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}\,.$$

[2]see footnote 2 on p.8 of the week 3 lecture notes of Calculus 1 - you need to have read Thomas' Calculus Section 2.3 to fully appreciate this definition!

We need to avoid the whole path to the limit where $x = y$, hence the condition $x \neq y$. Accordingly, there is a problem with just setting $x = y = 0$ because $\sqrt{x} - \sqrt{y} \to 0$ as $(x, y) \to (0, 0)$. However, we can write

$$\begin{aligned}
\lim_{(x,y)\to(0,0),x\neq y} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y)\to(0,0),x\neq y} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \\
&= \lim_{(x,y)\to(0,0),x\neq y} \frac{x(x-y)(\sqrt{x} + \sqrt{y})}{(x-y)} \\
&= \lim_{(x,y)\to(0,0),x\neq y} x(\sqrt{x} + \sqrt{y}) = 0\,.
\end{aligned}$$

Now we use limits to define continuity for a function of two variables.

Reminder: Continuity

For functions of one variable $f(x)$ is **continuous** at $x = a$ whenever $f(a)$ is defined, $\lim_{x\to a} f(x)$ exists and the limit L equals $f(a)$, that is, $\lim_{x\to a} f(x) = f(a)$. Analogously:

DEFINITION Continuous Function of Two Variables

A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

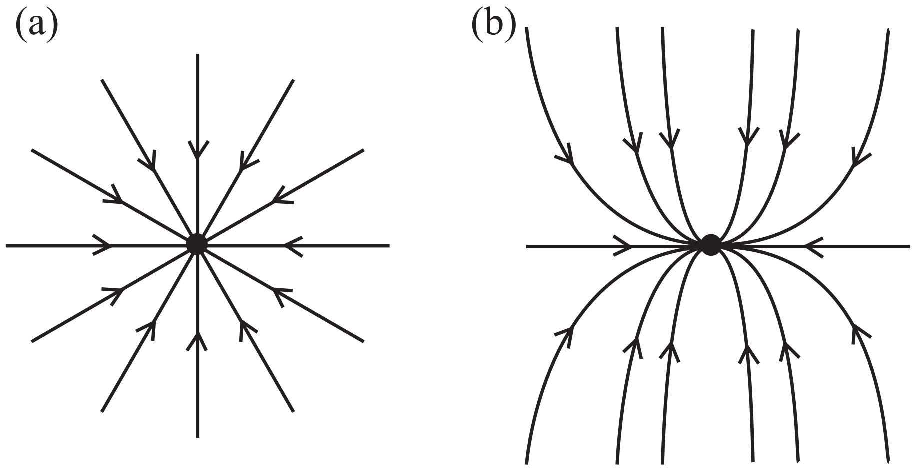
It follows from the previous Theorem that polynomials and rational functions of two variables are continuous on their domains.

Recall that for functions of one variable both the left- and the right-sided limits had to have *the same value* for a limit to exist at a point. For functions of two (or more) variables, this translates into the **Two-Path Test for Nonexistence of a Limit**: It states that if a function $f(x, y)$ has different limits along two different paths as $(x, y) \to (x_0, y_0)$, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

does not exist.

The following figure illustrates this concept for paths approaching a point in radial and tangential directions:



To have a limit at a point we have to have *the same limit* as the point is approached from all directions, including (a) radial directions and (b) tangential directions.

Example:

Show that the function

$$f(x,y) = \frac{2x^2y}{x^4+y^2}$$

has no limit as $(x,y) \to (0,0)$.

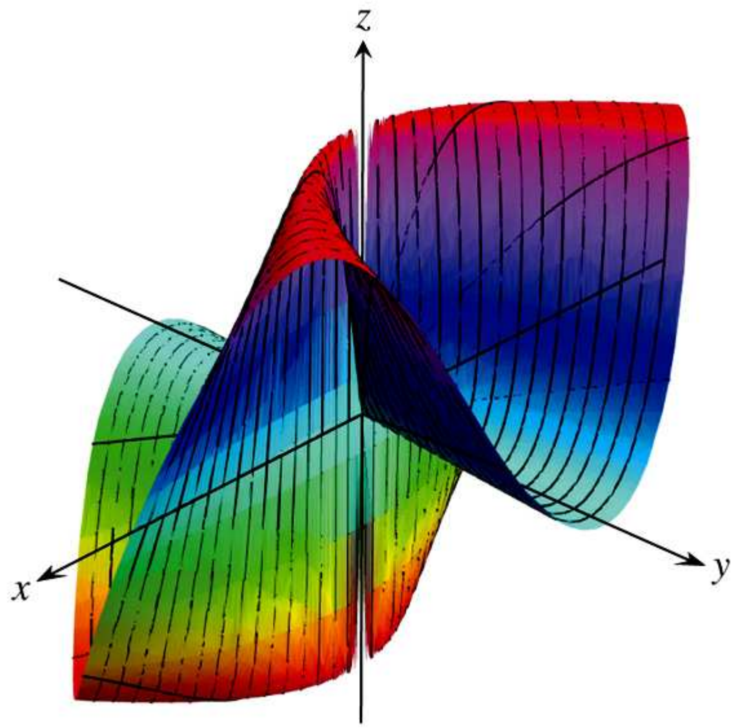
We cannot use substitution as it leads to $0/0$. However, we can consider what happens as we approach $(0,0)$ along a family of different curves. Remember, the choice of curves is up to us as the *Two-Path Test* does not specify what the path should be. You may wish to check, as an exercise, what happens for the family of paths $y = mx$ as $(x,y) \to (0,0)$. Here we consider the next more complicated case, which is the family of parabolas given by $y = kx^2$ $(x \neq 0)$. Along these curves the function is

$$f(x,y)\big|_{y=kx^2} = \left.\frac{2x^2y}{x^4+y^2}\right|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4+(kx^2)^2} = \frac{2kx^4}{x^4+k^2x^4} = \frac{2k}{1+k^2}\,.$$

Therefore, as we approach $(0,0)$ along any curve $y = kx^2$, we have

$$\lim_{(x,y)\to(0,0)} \left[f(x,y)\big|_{y=kx^2} \right] = \frac{2k}{1+k^2}\,.$$

Consequently, the actual limit depends on which path of approach we take (i.e. which parabola we are on which is determined by the value of k). By the *Two-Path Test* there is hence *no limit* as $(x,y) \to (0,0)$. This is illustrated by looking at the surface of this function:


z
x
y