Question 1 (17 marks) Let $\{X_t\}_{t=1,2,...}$ be a time series such that

$$X_t = m_t + s_t + Y_t,$$

where m_t denotes a polynomial trend of degree k, s_t denotes a seasonal effect with period length d and Y_t denotes a zero-mean stationary process with autocovariance function $\gamma_Y(\tau)$, $\tau = 0, \pm 1, \pm 2, \ldots$. It is assumed that $s_t = s_{t-d}$.

(a) Define the operators ∇ and ∇_d, and explain how they can be used to remove the trend and seasonality from the time series $\{X_t\}$. [6]

(b) Show that $\nabla_d X_t$ is a stationary process for $k = 1$ and give its autocovariance function. [5]

(c) Outline the main steps of the classical decomposition method for estimating the trend and seasonal effects. [6]

Question 2 (22 marks) Consider an MA(2) process of the form

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2},$$

where $\{Z_t\} \sim WN(0, \sigma^2)$.

(a) Show that the autocorrelation function of this process is given by

$$\rho(\tau) = \begin{cases} 1 & \text{if} \quad \tau = 0, \\ \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} & \text{if} \quad \tau = \pm 1, \\ \frac{\theta_2}{1+\theta_1^2+\theta_2^2} & \text{if} \quad \tau = \pm 2, \\ 0 & \text{if} \quad |\tau| > 2. \end{cases}$$

How does this function behave for an MA(q) process? [12]

(b) State a necessary and sufficient condition for the above MA(2) process to be invertible. For what values of θ_1 and θ_2 is the process invertible? [6]

(c) Define the seasonal MA$(2)_h$ process and give the equivalent condition for this process to be invertible. [4]

Question 3 (24 marks) Let the causal process for a time series $\{X_t\}$ be given by

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t + \theta Z_{t-1},$$

where $\{Z_t\} \sim WN(0, \sigma^2)$.

(a) Write down the operator form of this process. Under what conditions would this be an ARMA(2, 1) process? [4]

(b) Obtain the linear process form of this time series when $\phi_1 = 0.3$, $\phi_2 = 0.4$ and $\theta = 0.9$. [15]

(c) State the difference equations in terms of the autocorrelation function for an ARMA(2, 1) process. How does this function behave for this process? [5]