3: VECTOR DIFFERENTIATION, ∇, GRAD, DIV AND CURL.

3.1 Vector Functions of 1 or more variables.

We have already met a curve $\underset{\sim}{r}(t)$ and its tangent $\dfrac{d\underset{\sim}{r}}{dt}$

and $\underset{\sim}{r}(u,v)$ and two tangent vectors $\dfrac{\partial \underset{\sim}{r}}{\partial u}, \dfrac{\partial \underset{\sim}{r}}{\partial v}$.

We can easily extend this to a general vector $\underset{\sim}{F}$ which isn't a position vector:

eg if $\underset{\sim}{F}(t)$ depends ~~on~~ on one variable t,

$$\underset{\sim}{F}(t) \equiv (F_1(t), F_2(t), F_3(t))$$

$$\equiv F_1(t)\underset{\sim}{i} + F_2(t)\underset{\sim}{j} + F_3(t)\underset{\sim}{k}$$

then we can define $\dfrac{d\underset{\sim}{F}}{dt}$ in the obvious way as

$$\frac{d\underset{\sim}{F}}{dt} = \left(\frac{dF_1}{dt}, \frac{dF_2}{dt}, \frac{dF_3}{dt} \right).$$

Also if $\underset{\sim}{F}(t)$ and $\underset{\sim}{G}(t)$ are two vector functions of t, then

$$\frac{d}{dt}(\underset{\sim}{F} \cdot \underset{\sim}{G}) = \underset{\sim}{F} \cdot \frac{d\underset{\sim}{G}}{dt} + \frac{d\underset{\sim}{F}}{dt} \cdot \underset{\sim}{G}.$$

Proof: $\underset{\sim}{F} \cdot \underset{\sim}{G} = F_1 G_1 + F_2 G_2 + F_3 G_3$

$$\frac{d}{dt}(\underset{\sim}{F} \cdot \underset{\sim}{G}) = F_1 \frac{dG_1}{dt} + \frac{dF_1}{dt} G_1 + \cdots$$

$$= \underset{\sim}{F} \cdot \frac{d\underset{\sim}{G}}{dt} + \frac{d\underset{\sim}{F}}{dt} \cdot \underset{\sim}{G}.$$

A similar result applies for cross product.

3.2 VECTOR FIELDS.

Defn: A vector field is a vector whose value depends on position in 3-D, i.e x, y, z.

May also vary with time t.

$$\underset{\sim}{F} \equiv \underset{\sim}{F}(x,y,z) \equiv \underset{\sim}{F}(\underset{\sim}{r})$$

$$\equiv \left(F_1(x,y,z),\ F_2(x,y,z),\ F_3(x,y,z)\right)$$

$$\equiv F_1(x,y,z)\,\underset{\sim}{i} + F_2(x,y,z)\,\underset{\sim}{j} + F_3(x,y,z)\,\underset{\sim}{k}.$$

Physical examples:

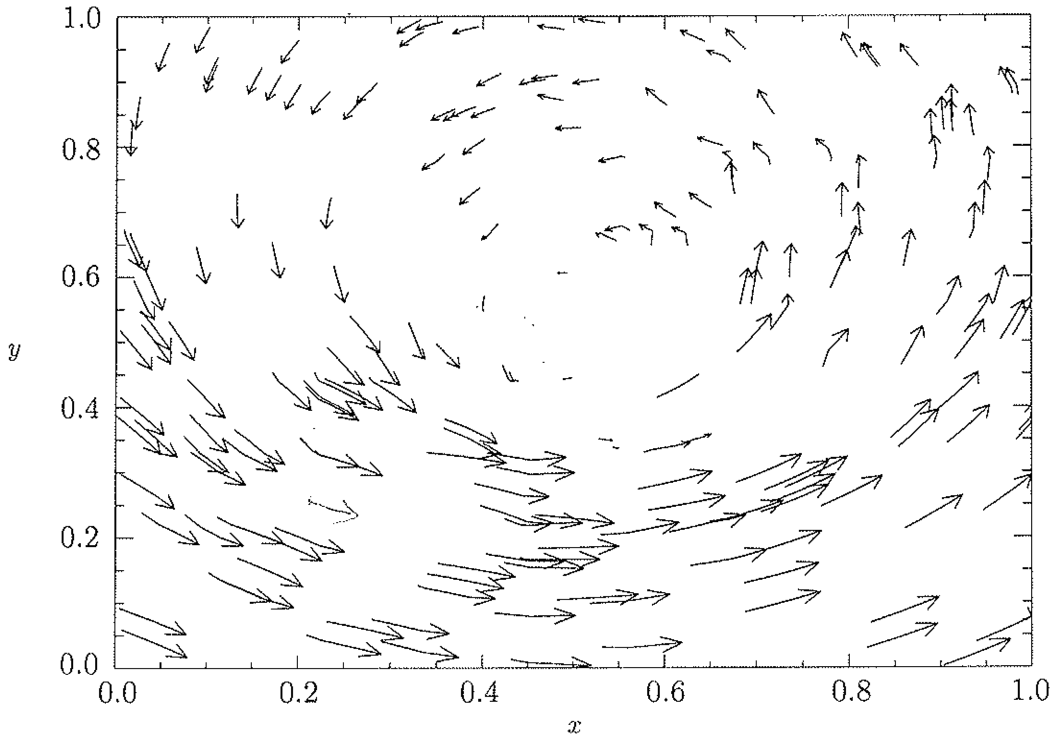
Velocity in a fluid

Magnetic field.

We can add vector fields in the obvious way: if $\underset{\sim}{F}(\underset{\sim}{r})$ and $\underset{\sim}{G}(\underset{\sim}{r})$ are two vector fields, so is $\underset{\sim}{F} + \underset{\sim}{G}$.

Multiplication by a constant:

$\lambda \underset{\sim}{F}$ is also a vector field.



1.0
0.8
0.6
0.4
0.2
0.0
y
0.0
0.2
0.4
0.6
0.8
1.0
x

Differentiating a vector field :

Given $\underset{\sim}{F}(\underset{\sim}{r}) = F_1(x,y,z)\,\underset{\sim}{i} + F_2(x,y,z)\,\underset{\sim}{j} + F_3(x,y,z)\,\underset{\sim}{k}$,

we have a total of 9 partial derivatives

eg $\dfrac{\partial F_1}{\partial x}$, $\dfrac{\partial F_2}{\partial x}$, ... $\dfrac{\partial F_3}{\partial z}$.

These form a "3×3 matrix" called a <u>tensor</u>. These are beyond the scope of this course.

We will now take <u>combinations</u> of these to give a scalar field and a vector field :

recall the operator

$$\underset{\sim}{\nabla} \equiv \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z} .$$

To get a well-behaved answer, we need to take the scalar or vector product of $\underset{\sim}{\nabla}$ with $\underset{\sim}{F}$.

3.3 DIVERGENCE OF A VECTOR FIELD.

Suppose $\underset{\sim}{F}(x,y,z) \equiv F_1\,\underset{\sim}{i} + F_2\,\underset{\sim}{j} + F_3\,\underset{\sim}{k}$

is a vector field.

The divergence of $\underset{\sim}{F}$ is defined to be

$$\operatorname{div} \underset{\sim}{F} \equiv \underset{\sim}{\nabla} \cdot \underset{\sim}{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

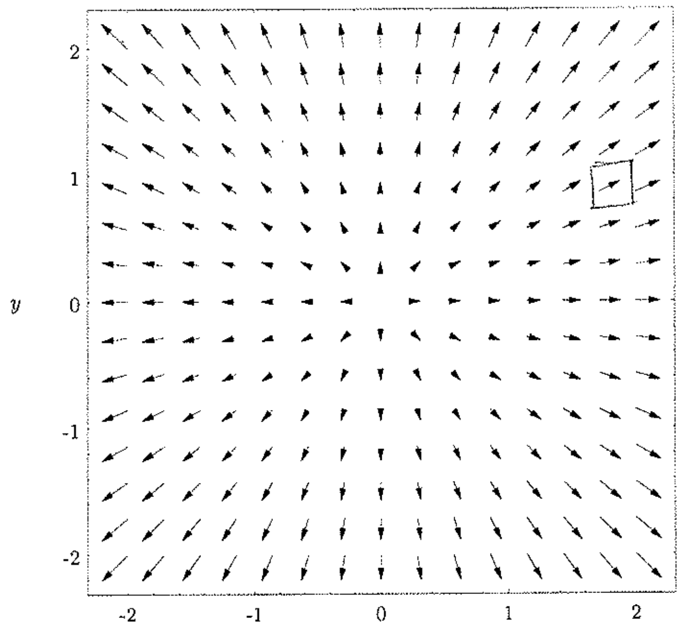
$\underset{\sim}{\nabla} \cdot \underset{\sim}{F}$ is a <u>scalar</u> and depends on position (x,y,z), so it is a <u>scalar field</u>.

We can also get the above by writing out

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right) \cdot \left(F_1\,\underset{\sim}{i} + F_2\,\underset{\sim}{j} + F_3\,\underset{\sim}{k}\right)$$

$$= \underbrace{(\underset{\sim}{i}\cdot\underset{\sim}{i})}_{1}\frac{\partial F_1}{\partial x} + \underbrace{(\underset{\sim}{i}\cdot\underset{\sim}{j})}_{0}\frac{\partial F_2}{\partial x} + \cdots + \underbrace{(\underset{\sim}{k}\cdot\underset{\sim}{k})}_{1}\frac{\partial F_3}{\partial z}$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$



2
1
y
0
-1
-2
-2
-1
0
1
2

Example 3.3:

If $\underset{\sim}{F} = 3xy^2\,\underset{\sim}{i} + e^z\,\underset{\sim}{j} + xy\sin z\,\underset{\sim}{k}$,
calculate $\underset{\sim}{\nabla} \cdot \underset{\sim}{F}$.

Answer: $$\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = \frac{\partial}{\partial x}\left(3xy^2\right) + \frac{\partial}{\partial y}\left(e^z\right) + \frac{\partial}{\partial z}\left(xy\sin z\right)$$

$$= 3y^2 + 0 + xy\cos z.$$

Easy to show that if $\underset{\sim}{F}, \underset{\sim}{G}$ are two vector fields and λ is a constant,

$$\underset{\sim}{\nabla} \cdot \left(\underset{\sim}{F} + \underset{\sim}{G}\right) = \underset{\sim}{\nabla} \cdot \underset{\sim}{F} + \underset{\sim}{\nabla} \cdot \underset{\sim}{G}$$

$$\underset{\sim}{\nabla} \cdot \left(\lambda \underset{\sim}{F}\right) = \lambda\left(\underset{\sim}{\nabla} \cdot \underset{\sim}{F}\right)$$

Note: only true if λ is <u>constant</u>, indep. of x, y, z.
If λ is a scalar field, see 3.5.

Defn:

If $\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 0$ (in some region)

then $\underset{\sim}{F}$ is called DIVERGENCE-FREE

or SOLENOIDAL.

3.4 CURL OF A VECTOR FIELD.

Given a vector field $\underset{\sim}{F}(\underset{\sim}{r})$

i.e. $\underset{\sim}{F}(\underset{\sim}{r}) = F_1(\underset{\sim}{r})\,\underset{\sim}{i} + F_2(\underset{\sim}{r})\,\underset{\sim}{j} + F_3(\underset{\sim}{r})\,\underset{\sim}{k}$,

the curl of $\underset{\sim}{F}$ is defined as

$$\text{curl}\,\underset{\sim}{F} \equiv \underset{\sim}{\nabla} \times \underset{\sim}{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\underset{\sim}{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\underset{\sim}{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underset{\sim}{k}$$

Note $\text{curl}\,\underset{\sim}{F}$ is a vector field, since there are $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ on the RHS and it generally depends on x, y, z.

Also :

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Write out in full $\rightarrow$ original def.

Looks like 'cross product' of $\underset{\sim}{\nabla}$ with $\underset{\sim}{F}$.

Example:

Vector Field $\underset{\sim}{V} = y\underset{\sim}{i} - x\underset{\sim}{j}$.

Find $\underset{\sim}{\nabla} \times \underset{\sim}{V}$.

Answer:
$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{vmatrix}$$

$$= \underset{\sim}{i}\,(0-0) + \underset{\sim}{j}\,(0-0) + \underset{\sim}{k}\,(-1-1)$$

$$= -2\underset{\sim}{k}.$$

Note: here $\underset{\sim}{\nabla} \times \underset{\sim}{V}$ is a constant vector

But usually it is not const.

normal to **n**, the vector field tends to go round in an anticlockwise direction if one looks along vector **n**. If the component of the curl were negative, it would mean that the vector field tends to go round in a clockwise direction. (See Fig. 3.4.) This idea will be made more precise when we come to Stokes's Theorem.

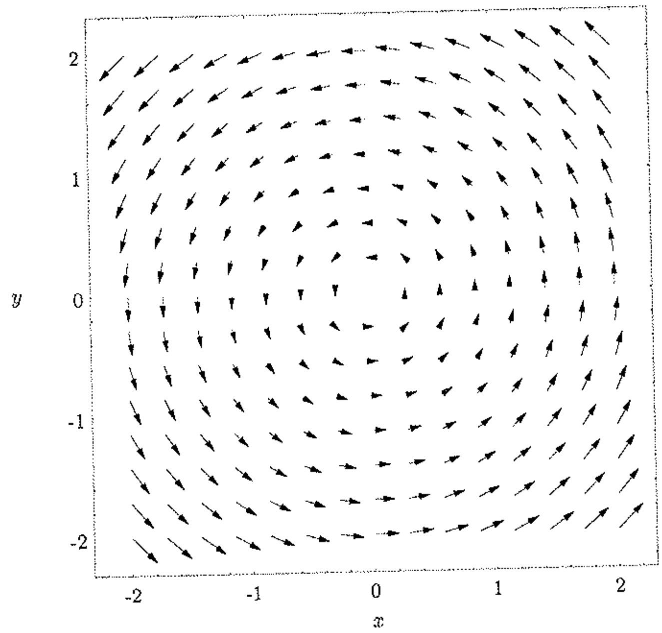


Figure 3.4: Example of a vector field with positive curl (in the z direction): $\mathbf{F} = x\mathbf{j} - y\mathbf{i}$.

Exercise 3.3 :

$$\underset{\sim}{F} = (x^2+y^2+z^2)\,\underset{\sim}{i} + (x^4 - y^2 z^2)\,\underset{\sim}{j} + xyz\,\underset{\sim}{k} .$$

Find $\underset{\sim}{\nabla} \times \underset{\sim}{F}$.

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2+z^2 & x^4-y^2z^2 & xyz \end{vmatrix}$$

$$= \underset{\sim}{i}\left(xz - (-2zy^2)\right) + \underset{\sim}{j}\left(2z - yz\right) + \underset{\sim}{k}\left(4x^3 - 2y\right)$$

Definition :

If $\underset{\sim}{\nabla} \times \underset{\sim}{F} = \underset{\sim}{0}$ throughout some volume,

then $\underset{\sim}{F}$ is called CURL-FREE

or IRROTATIONAL

in that volume.

As for grad & div, curl obeys the 'obvious' rules for addition and multiplication by a constant λ :

i.e. $\underset{\sim}{\nabla} \times (\underset{\sim}{F} + \underset{\sim}{G}) = \underset{\sim}{\nabla} \times \underset{\sim}{F} + \underset{\sim}{\nabla} \times \underset{\sim}{G}$

and $\underset{\sim}{\nabla} \times (\lambda \underset{\sim}{F}) = \lambda (\underset{\sim}{\nabla} \times \underset{\sim}{F})$.

Meaning of curl $\underset{\sim}{F}$:

Roughly, curl $\underset{\sim}{F}$ measures the tendency of $\underset{\sim}{F}$ to 'spin' around given point $\underset{\sim}{r}$.

Given a unit vector $\underset{\sim}{n}$,
if $(\text{curl } \underset{\sim}{F}) \cdot \underset{\sim}{n}$ is positive (negative) then $\underset{\sim}{F}$ tends to spin clockwise (anticlockwise) looking along direction $\underset{\sim}{n}$.

$$\left|\begin{array}{ccc|cc} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 & F_3 & F_1 & F_2 \end{array}\right.$$

Green +

Red −.