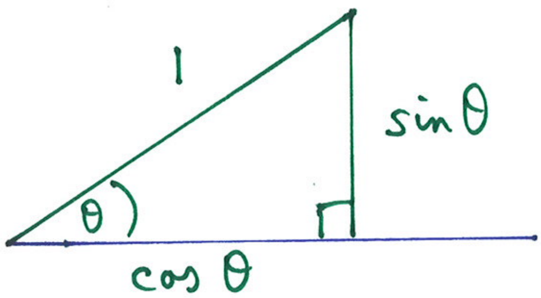
1.1 Trigonometric functions.



Special values:

	$\theta = 0$	$\frac{\pi}{6} = 30^\circ$	$\frac{\pi}{4} = 45^\circ$	$\frac{\pi}{3} = 60^\circ$	$\frac{\pi}{2} = 90^\circ$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

Which are positive?

'Quadrant'	θ in rad	Sin	Cos	Tan	Positive
1	$0 < \theta < \frac{\pi}{2}$	+	+	+	All
2	$\frac{\pi}{2} < \theta < \pi$	+	−	−	Sin
3	$\pi < \theta < \frac{3\pi}{2}$	−	−	+	Tan
4	$\frac{3\pi}{2} < \theta < 2\pi$	−	+	−	Cos

'Add Sugar To Coffee' rule.

Also $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$

$$\cos\left(x+\frac{\pi}{2}\right) = -\sin x, \quad \sin\left(x+\frac{\pi}{2}\right) = \cos x$$

$$\cos(x+\pi) = -\cos x \quad \sin(x+\pi) = -\sin x.$$

(1.1).

$\rightarrow$ $\cos(n\pi) = (-1)^n$, $\sin(n\pi) = 0$ etc

$$\sin\left(\left(n+\tfrac{1}{2}\right)\pi\right) = (-1)^n \quad (1.4).$$

(for n = integer)

MEMORISE:

$$\left\{\begin{array}{l} \sin^2 A + \cos^2 A = 1 \\ \cos(A+B) = \cos A \cos B - \sin A \sin B \\ \sin(A+B) = \sin A \cos B + \cos A \sin B \end{array}\right.$$

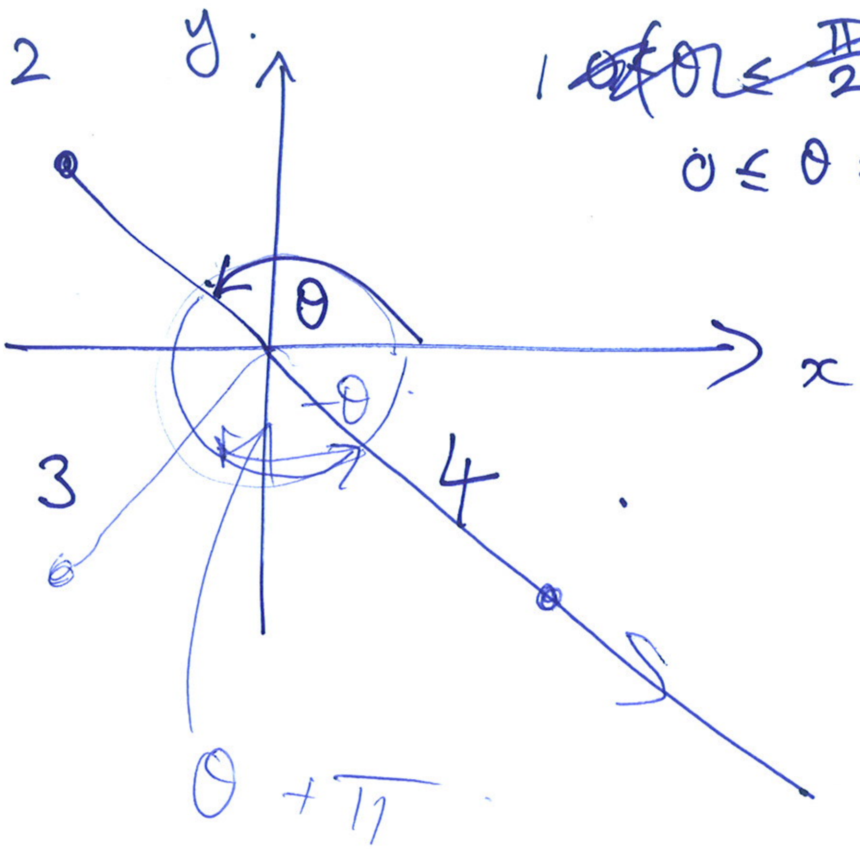
(1.5 – 1.7).

$$\begin{aligned} \cos(A-B) &= \cos(A+(-B)) \\ &= \cos A \cos(-B) - \sin A \sin(-B) \\ &= \cos A \cos B + \sin A \sin B \end{aligned}$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B.$$

$1 \cancel{(\theta \leq \frac{\pi}{2})}$

$0 \leq \theta \leq \frac{\pi}{2}$.



$$\cos 2A = \cos^2 A - \sin^2 A$$

$$= (1 - \sin^2 A) - \sin^2 A$$

$$= 1 - 2\sin^2 A .$$

$$\sin 2A = 2 \sin A \cos A .$$

Let $y = 2A$, $A = \frac{1}{2} y$

$$\sin y = 2 \sin \tfrac{1}{2} y \cos \tfrac{1}{2} y .$$

$$\sin^2 A = \frac{1}{2}\left[1 - \cos 2A\right] .$$

$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B.$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]. \quad (1.9).$$

$$\frac{d}{dx}(\sin(kx)) = k\cos(kx)$$

(k constant)

$$\frac{d}{dx}(\cos(kx)) = -k\sin(kx).$$

$$\hookrightarrow \quad \frac{d^2}{dx^2}(\sin kx) = -k^2 \sin kx$$

$$\frac{d}{dx^2}(\cos kx) = -k^2 \cos kx.$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -k^2 y$$

$$\Leftrightarrow \quad y = a\cos kx + b\sin kx$$

What about?

$$\frac{d^2y}{dx^2} = +k^2 y \quad ?$$

$$\Rightarrow \quad y = a\cosh kx + b\sinh kx.$$

1.2 ln, exp, and hyperbolic functions.

Definition : $\ln x \equiv \int_1^x \frac{dt}{t}$ $(x>0)$

$$\Rightarrow \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Properties : $\ln(ab) = \ln a + \ln b$

$\ln(x^p) = p \ln x$

$$\Rightarrow \ln(1/a) = -\ln a$$

$$\ln(a/b) = \ln a - \ln b.$$

$$\ln 1 = 0$$

Definition :

$\exp x$ is inverse fn. to $\ln x$

so $\exp(\ln x) = \ln(\exp x) = x.$

$$\exp 1 = e = 2.718\ldots$$

$$\exp x = e^x.$$

$$\Rightarrow \exp(a+b) = e^{a+b} = e^a e^b.$$

$$\frac{d}{dx}(\exp x) = \exp x.$$

$$\frac{d}{dx}(\exp(kx)) = k\exp(kx)$$

Definitions : $\cosh x \equiv \frac{1}{2}\left(e^{x} + e^{-x}\right)$

$$\sinh x \equiv \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\cosh^2 x - \sinh^2 x = 1$$

Addition formulae 'similar' to sin, cos.

$$\frac{d}{dx}(\sinh kx) = k \cosh kx$$

$$\frac{d}{dx}(\cosh kx) = k \sinh kx$$

so $$\frac{d^2y}{dx^2} = +k^2 y$$

has solution $$y = a \cosh kx + b \sinh kx \qquad (1.13).$$

$$\frac{d^2y}{dx^2} = cy \, . \qquad (c \overset{?}{=} \text{const}) \, .$$

$c > 0$:

let $k = \sqrt{c}$.

$$y = a\cosh kx + b\sinh kx$$

$c = 0$

$$y = ax + b \, .$$

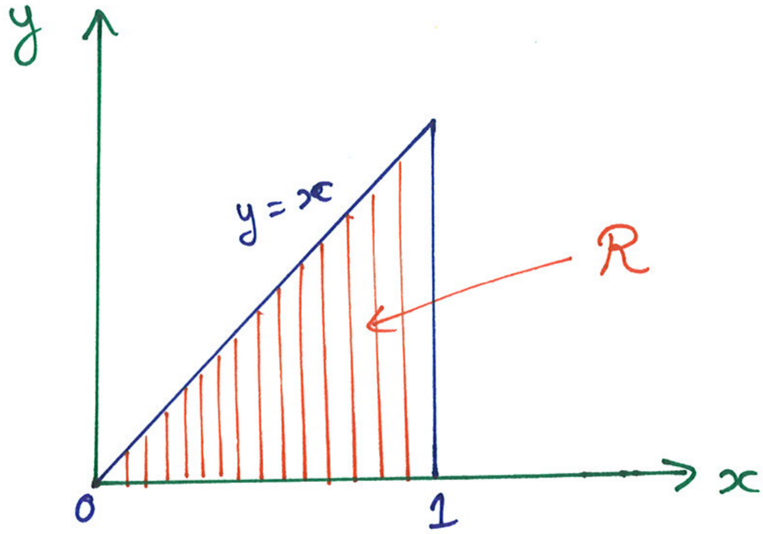
$c < 0$

let $k = \sqrt{-c}$.

$$y = a\cos kx + b\sin kx \, .$$

1.3 DOUBLE & TRIPLE INTEGRALS.

Example:



Given $f(x,y) = x^2y^2$,

evaluate $\int_R f(x,y)\, dA$ over triangular region R.

R is given by $0 \le x \le 1$, $0 \le y \le x$.

$dA = dx\, dy$ (area in xy plane).

$\rightarrow$ want $\displaystyle\int_{x=0}^{1}\left(\int_{y=0}^{x} x^2y^2 \, dy\right) dx$.

$$= \int_{x=0}^{1} x^2 \underbrace{\left(\int_{y=0}^{x} y^2 \, dy\right)}_{=\left[\frac{1}{3}\cancel{x}y^3\right]_0^x \;=\; \frac{1}{3}x^3} dx$$

$$= \int_{x=0}^{1} x^2 . \left(\frac{1}{3}x^3\right) dx$$

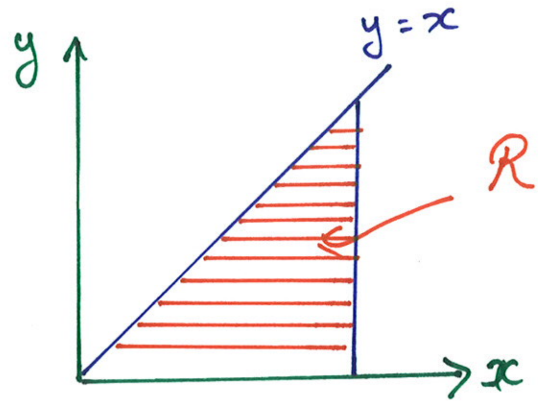
$$= \left[\frac{1}{18}x^6\right]_0^1 = \frac{1}{18} .$$

Careful on order :

$$\int_{y=0}^{x} \left(\int_{x=0}^{1} x^2 y^2 \, dx \right) dy = \frac{x^3}{9}$$

~~~~ WRONG -
answer can't depend on x.

We can do 'y integral outside',
but need to change limits ...



'dy outside' $\Rightarrow$ y limits must be the
full range of R, x - lims depend on y.

$\Rightarrow \quad 0 \le y \le 1, \quad y \le x \le 1.$

want $\displaystyle\int_{y=0}^{1} \left(\int_{x=y}^{1} x^2 y^2 \, dx \right) dy$

$$= \int_{y=0}^{1} y^2 \underbrace{\left(\int_{x=y}^{1} x^2 \, dx \right)}_{\left[\frac{1}{3}x^3\right]_y^1 = \frac{1}{3} - \frac{1}{3}y^3.} dy$$
~~~~

$$= \int_{y=0}^{1} y^2 \left(\frac{1}{3} - \frac{1}{3}y^3\right) \, dy$$

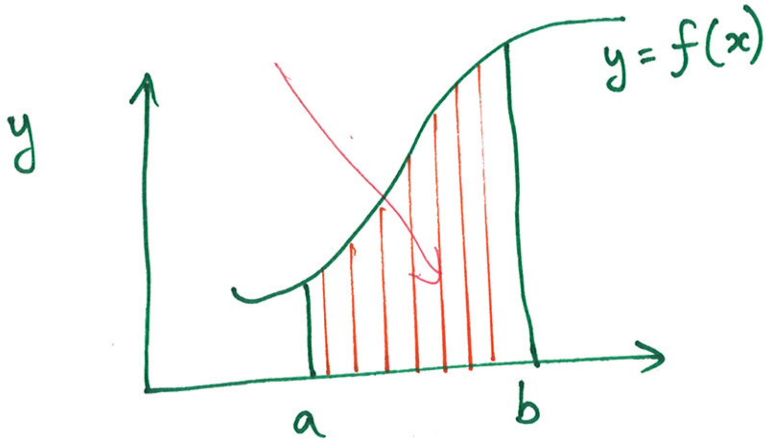
$$= \int_{y=0}^{1} \frac{1}{3}y^2 - \frac{1}{3}y^5 \, dy$$

$$= \left[\frac{1}{9}y^3 - \frac{1}{18}y^6\right]_0^1$$

$$= \frac{1}{9} - \frac{1}{18} = \frac{1}{18}.$$

Geometrical Picture :

$\boxed{1D}$: $\int_a^b f(x)\, dx$ = Area

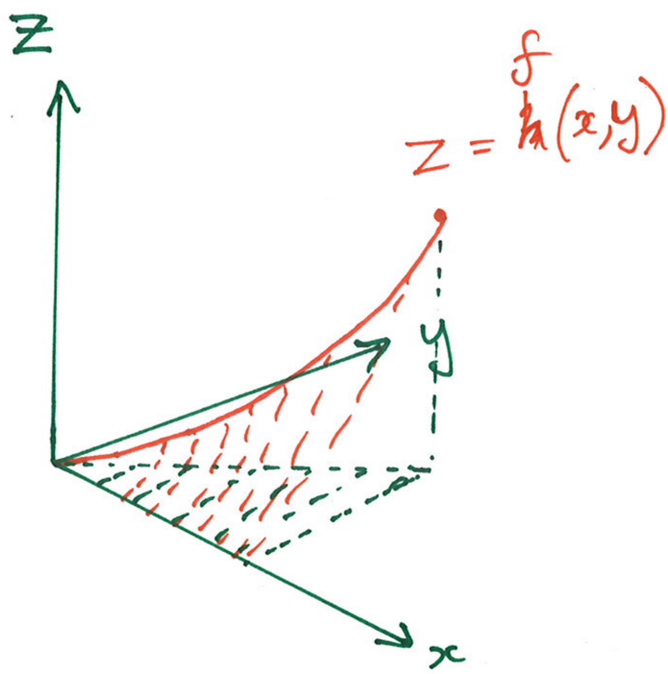


$\boxed{2D:}$ $\iint_R f(x,y)\, dx\, dy$

= volume between surface

$z = f(x,y)$

and $z=0$ plane, above region R.



$\boxed{3D}$

VOLUME INTEGRALS :

$$\int_V f(x,y,z)\ dV$$

$$= \iiint_V \left(f(x,y,z)\ dx \right) dy\ dz$$

= 4-D 'hypervolume'

if $f(x,y,z)$ pokes into the 4th dimension.

or e.g.

if $f(x,y,z)$ = Density

$= \rho(x,y,z)$

= Mass / unit volume

$$\int_V \rho(x,y,z)\ dV = \int_V dM$$

= Mass inside volume V.

Changing coordinates :

$$\iiint f(x,y,z)\, dx\, dy\, dz$$

$$= \iiint f(u,v,w)\, J \, du\, dv\, dw.$$

Jacobian .

$$J(x,y,z\,;\,u,v,w) = \det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \cdots \\ - & - & - - \end{vmatrix}$$

1.4 CURVES & SURFACES.

Curves in 2D :

3 ways to specify :

i) $y = f(x)$ "explicit form"

ii) $g(x,y) = \text{constant}$ "implicit form"

iii) $x = f(t),\ y = g(t)$ "parametric form".

See Chapter 2.

Converting (i) to (ii) is easy :

define $g(x,y) \equiv y - f(x)$

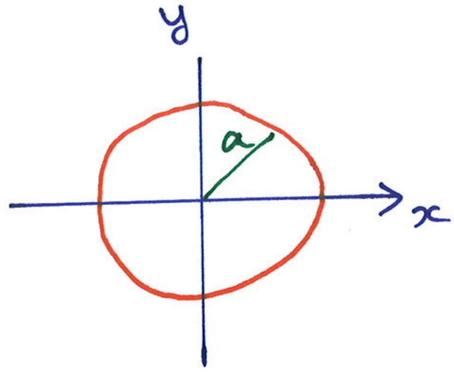
then $g(x,y) = 0$.

Conic sections in 2D :

- functions involve x^2, y^2, (xy) but no cubic or higher powers.
- 3~~4~~ types : ellipse ($\rightarrow$ circle)
  parabola
  hyperbola.

Circle : $x^2 + y^2 = a^2$

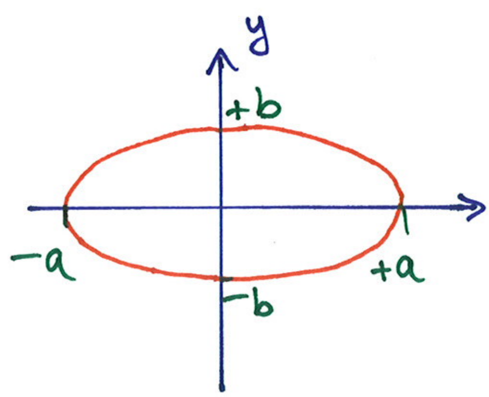
or $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1$



Circle centred at (b, c) of radius a :

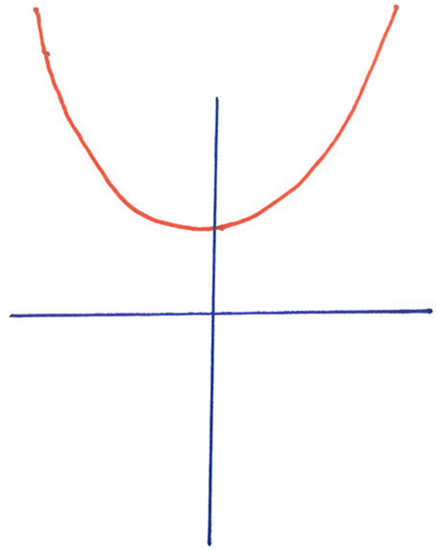
$$\frac{(x-b)^2}{a^2} + \frac{(y-c)^2}{a^2} = 1 \; .$$

Ellipse : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



Parabola: $y = ax^2 + b$

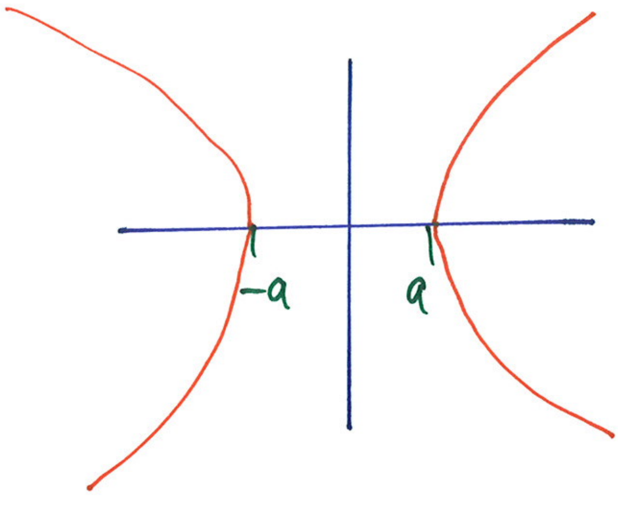
or $\frac{y}{a} - x^2 = \frac{b}{a}$



Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

or $cx^2 - ky^2 = d$, $ck > 0$

(note x^2 and y^2 coeffs. have opposite signs.).



1.5 SURFACES IN 3D.

$f(x, y) = \text{const} \rightarrow$ curve in 2D.

$f(x, y, z) = \text{const} \rightarrow$ surface in 3D.

$ax + by + cz = d \rightarrow$ plane in 3D.

Look at functions with x^2, y^2, z^2 (no cubics):

$$\left.\begin{aligned} x^2 + y^2 + z^2 &= a^2 \\ \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} &= 1 \end{aligned}\right\} \text{Sphere. radius } a.$$

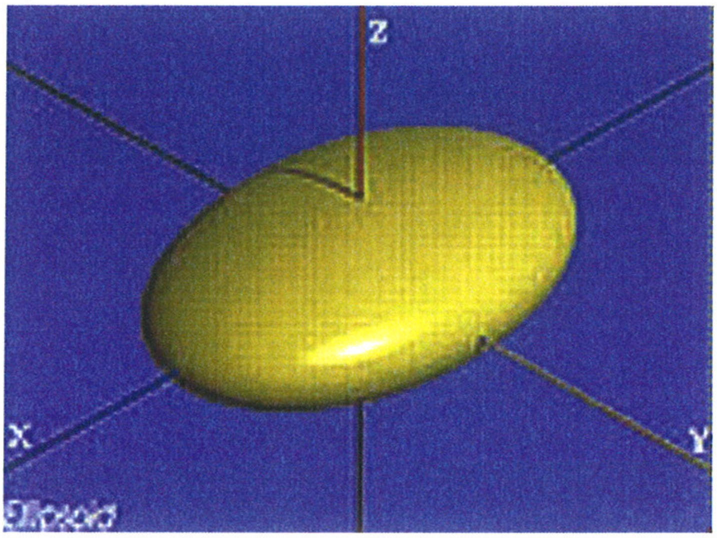
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{Ellipsoid}$$

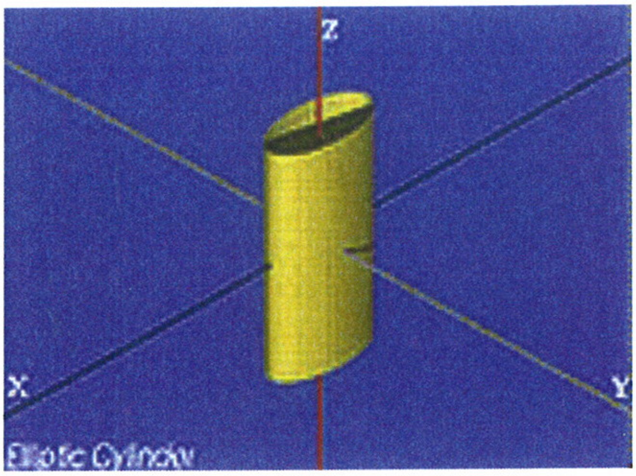
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\hookrightarrow$ z is missing

Ellipse in x, y plane, same at any plane $z = \text{const}$.

$\rightarrow$ Elliptical cylinder.


Z
X
Y
Ellipsoid


Z
X
Y
Elliptic Cylinder

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad ?$$

Consider 'slicing' with a plane

x, y or z = constant.

In each plane we get a curve, which we recognize.

'Join up' the slices.

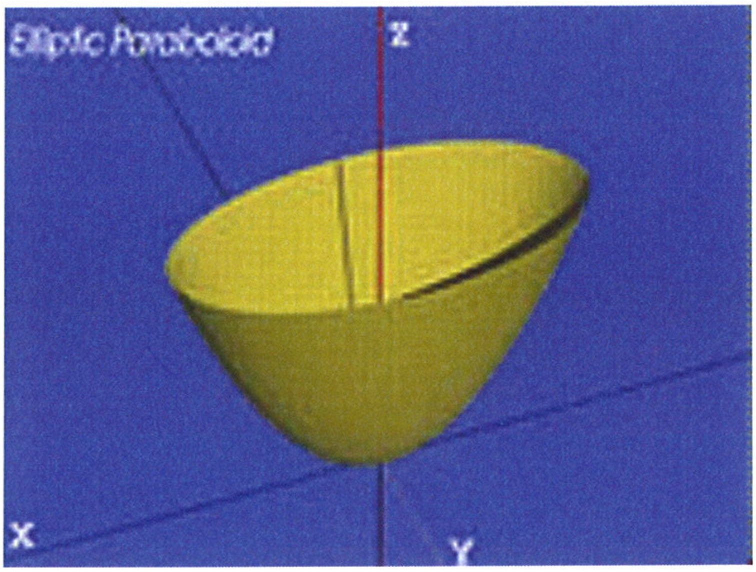
Plane $z = d$: $\begin{cases} d < 0 & - \text{ no solutions} \\ d = 0 & : x, y = (0,0). \\ d > 0 & : \text{ ellipse}. \end{cases}$

Plane $x = d$: $\frac{z}{c} = \frac{d^2}{a^2} + \frac{y^2}{b^2}$

$\rightarrow$ parabola.

$\Rightarrow$ surface is

ELLIPTIC PARABOLOID.


Elliptic Paraboloid
Z
X
Y

note

Consider $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = +1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z^2}{c^2}$$

$\rightarrow$ ellipse in any plane $z = d$.

hyperbola in any plane $x, y = d$.

$\Rightarrow$ "ELLIPTIC HYPERBOLOID (one sheet).

Finally $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} - 1 .$$

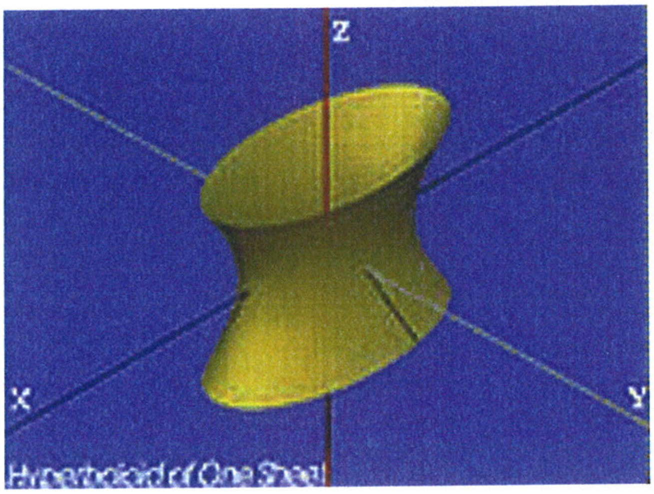
Plane $z = d$: if $|d| < c$,

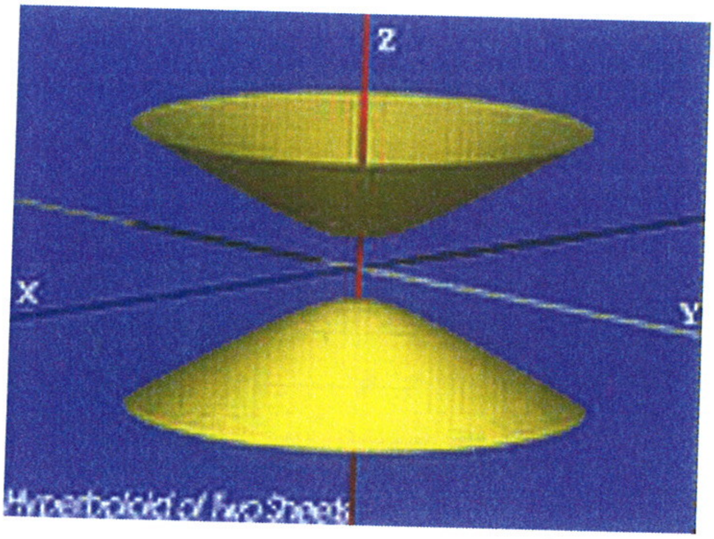
RHS negative, no solutions.

Ellipse if $|d| > c$.

Any plane $x, y = d$ $\rightarrow$ hyperbola.

$\Rightarrow$ ELLIPTIC HYPERBOLOID (two sheets).


Z
X
Y
Hyperboloid of One Sheet


Z
X
Y
Hyperboloid of Two Sheets

General case :

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Always a conic ; if $B \neq 0$
it will be 'tilted' from the
xy axes.

If $B = 0$, we can
'complete the square' :

eg $\quad x^2 + 6x + y^2 + 8y = 0$

$$\Rightarrow (x+3)^2 - 9 + (y+4)^2 - 16 = 0$$

$$(x+3)^2 + (y+4)^2 = 25$$

$\Rightarrow$ circle, radius 5,
centre $(-3, -4)$.

What about

$$Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + G = 0 \quad ?$$

i) ~~If~~ 'Complete squares' to get rid of D, E, F terms.

ii) Move constant to RHS and divide by it.

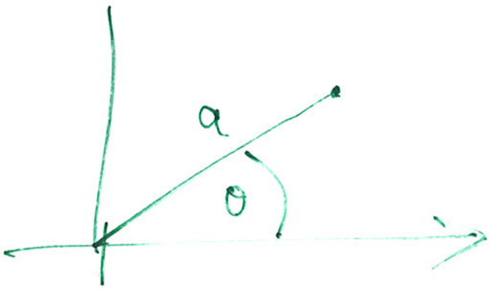
iii) If ~~necessary~~ two or three of x^2, y^2, z^2 coeffs are negative, mult. by -1.

→ get one of above standard forms

PARAMETRIC FORMS:

Circle :

$$x = a\cos\theta, \quad y = a\sin\theta .$$



$$x^2 + y^2 = a^2 .$$

Sphere :

$$\left.\begin{aligned} x &= a\sin\theta\cos\phi, \\ y &= a\sin\theta\sin\phi \\ z &= a\cos\theta \end{aligned}\right\} .$$

$$\rightarrow x^2 + y^2 + z^2 = a^2 .$$