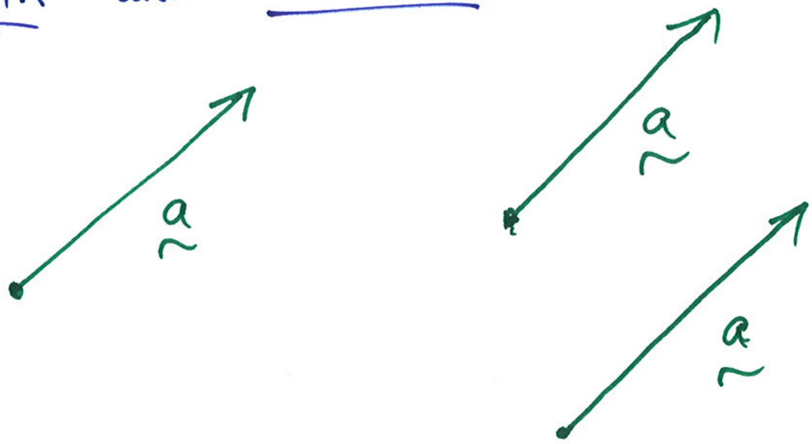
1.6 VECTORS.

A <u>vector</u> is a quantity with <u>length</u> and <u>direction</u>.



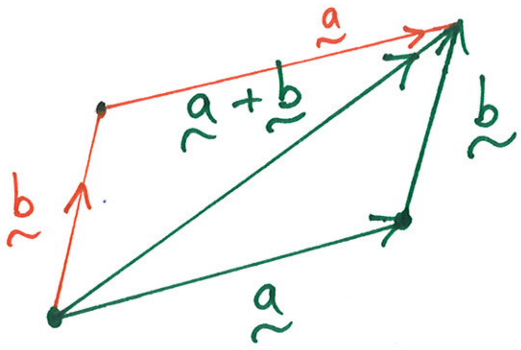
(Row) Vector $\underset{\sim}{v} = (v_1, v_2, v_3)$

$$\underset{\sim}{r} = (x, y, z)$$

$\underset{\sim}{r}$ always means this,
i.e. vector from origin to (x, y, z).

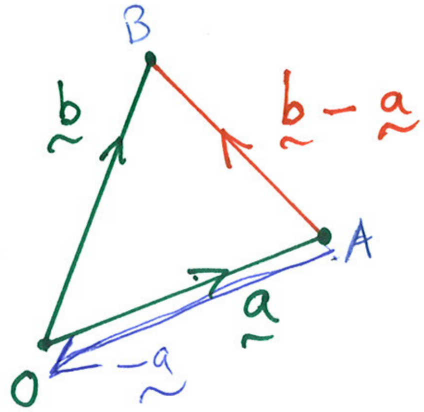
Length $|\underset{\sim}{v}| = \sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}}$

Addition of Vectors :



$$\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{b} + \underset{\sim}{a}$$

$$\underset{\sim}{a} + \underset{\sim}{b} = (a_1 + b_1 \, , \, a_2 + b_2 \, , \, a_3 + b_3) \, .$$

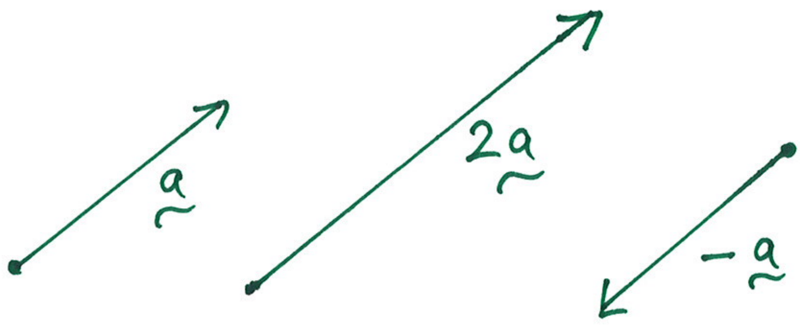


VECTOR <u>FROM</u>

$\underset{\sim}{a}$ <u>to</u> $\underset{\sim}{b}$ $= \underset{\sim}{b} - \underset{\sim}{a}$

Multiplication (by a scalar) :

$$\lambda \underset{\sim}{v} = (\lambda v_1 \, , \, \lambda v_2 \, , \, \lambda v_3) \, .$$



Define

$\underset{\sim}{i} = (1, 0, 0)$ = unit vector parallel to $+x$ axis

$\underset{\sim}{j} = (0, 1, 0)$ = " " " $+y$ axis

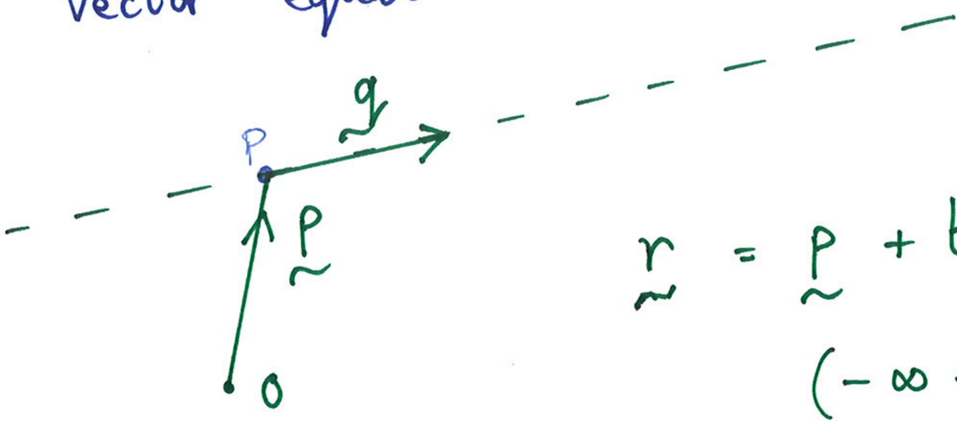
$\underset{\sim}{k} = (0, 0, 1)$ = " " " $+z$ axis.

so e.g. $\underset{\sim}{r} \equiv x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$\underset{\sim}{v} = v_1 \underset{\sim}{i} + v_2 \underset{\sim}{j} + v_3 \underset{\sim}{k}$

or (v_1, v_2, v_3)

Vector equation of a line:

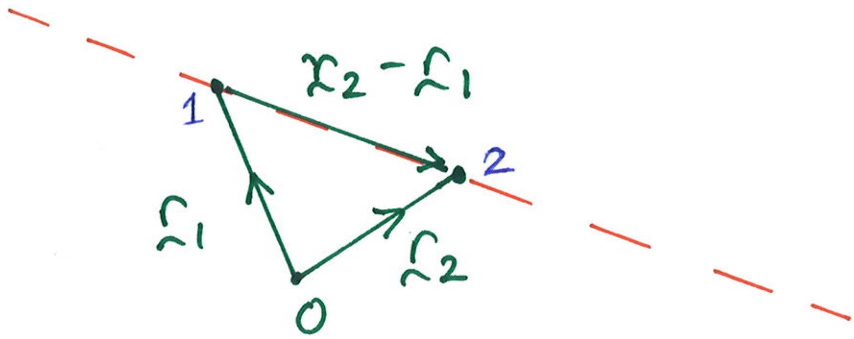


$$\underset{\sim}{r} = \underset{\sim}{p} + t\underset{\sim}{q}$$

$(-\infty < t < \infty)$

is straight line through point $\underset{\sim}{p}$ parallel to $\underset{\sim}{q}$.

Line through <u>two</u> given points :



$$\underset{\sim}{r} = \underset{\sim}{r}_1 + t(\underset{\sim}{r}_2 - \underset{\sim}{r}_1)$$

$-\infty < t < \infty$: infinite line

$0 < t < 1$: line segment from $\underset{\sim}{r}_1$ to $\underset{\sim}{r}_2$.

1.7 SCALAR & VECTOR PRODUCTS.

Scalar (or dot) product

$$\underset{\text{Vector}}{\underset{\sim}{a}} \cdot \underset{\text{Vector}}{\underset{\sim}{b}} = \underbrace{|\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos\theta}_{\text{Scalar.}}$$

In components,

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= \sum_{i=1}^{3} a_i b_i$$

In particular,

$$\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$$

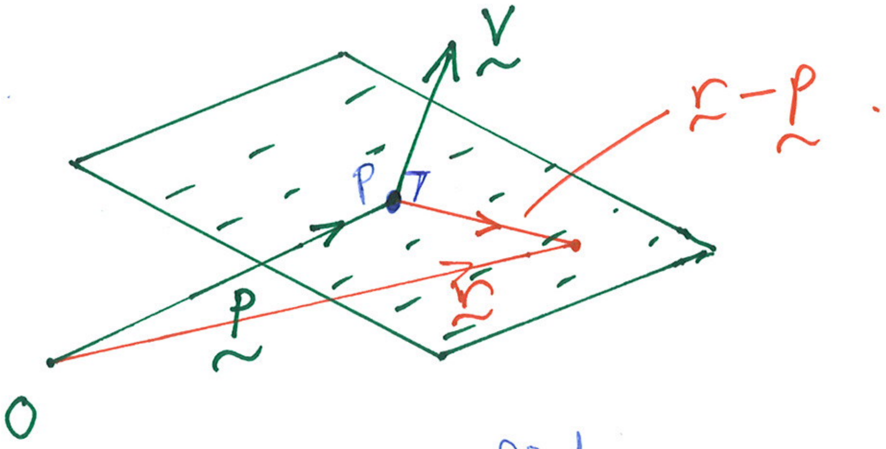
$$\underset{\sim}{a} \cdot \underset{\sim}{b} = 0 \iff \underset{\sim}{a} = \underset{\sim}{0},\ \underset{\sim}{b} = \underset{\sim}{0} \text{ or } \underset{\sim}{a} \perp \underset{\sim}{b}.$$

'Obvious' rules

eg $\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

$$(\lambda \underset{\sim}{a}) \cdot (\mu \underset{\sim}{b}) = \lambda \mu\, \underset{\sim}{a} \cdot \underset{\sim}{b}$$

Vector Equation of a Plane :



Consider plane containing fixed point $\underset{\sim}{p}$ and $\perp$ to vector $\underset{\sim}{v}$.

A general point $\underset{\sim}{r}$ is in this plane iff $\underset{\sim}{r} - \underset{\sim}{p} \perp \underset{\sim}{v}$

(or $\underset{\sim}{r} - \underset{\sim}{p} = \underset{\sim}{0}$).

$$\iff \quad (\underset{\sim}{r} - \underset{\sim}{p}) \cdot \underset{\sim}{v} = 0$$

$$\underset{\sim}{r} \cdot \underset{\sim}{v} - \underset{\sim}{p} \cdot \underset{\sim}{v} = 0$$

$$\underset{\sim}{r} \cdot \underset{\sim}{v} = \underbrace{\underset{\sim}{p} \cdot \underset{\sim}{v}}_{\text{constant for given } \underset{\sim}{p},\, \underset{\sim}{v}.}$$

$$v_1 x + v_2 y + v_3 z = \underset{\sim}{p} \cdot \underset{\sim}{v}$$
$$= d.$$

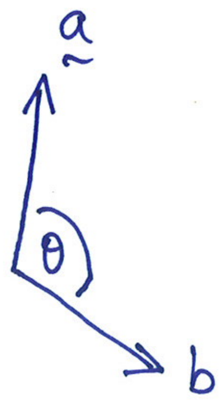
VECTOR PRODUCT :

The <u>vector product</u> of two vectors is a third vector :

$$\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta \; \hat{\underset{\sim}{n}}$$

Unit vector $\perp$ to $\underset{\sim}{a}, \underset{\sim}{b}$.

$$|\underset{\sim}{a} \times \underset{\sim}{b}| = \text{Area of parallellogram with sides } \underset{\sim}{a}, \underset{\sim}{b} \text{ .}$$



$\hat{\underset{\sim}{n}}$ points 'into the paper'.

Note this minus!

Rules :

$$\underset{\sim}{v} \times \underset{\sim}{w} = - \underset{\sim}{w} \times \underset{\sim}{v}$$

$$(\lambda \underset{\sim}{v}) \times \underset{\sim}{w} = \lambda(\underset{\sim}{v} \times \underset{\sim}{w}) = \underset{\sim}{v} \times (\lambda \underset{\sim}{w})$$

$$\underset{\sim}{u} \times (\underset{\sim}{v} + \underset{\sim}{w}) = \underset{\sim}{u} \times \underset{\sim}{v} + \underset{\sim}{u} \times \underset{\sim}{w} \text{ .}$$

$$\underset{\sim}{v} \times \underset{\sim}{v} = \underset{\sim}{0}$$ For any $\underset{\sim}{v}$.

$$\underset{\sim}{v} \times \underset{\sim}{w} = \underset{\sim}{0}$$

$\iff$ $\underset{\sim}{v}$ or $\underset{\sim}{w} = \underset{\sim}{0}$, or $\underset{\sim}{v}, \underset{\sim}{w}$ parallel.
(or antiparallel)

In components :

$$\underset{\sim}{v} \times \underset{\sim}{w} = \left(v_1 \underset{\sim}{i} + v_2 \underset{\sim}{j} + v_3 \underset{\sim}{k} \right) \times \left(w_1 \underset{\sim}{i} + w_2 \underset{\sim}{j} + w_3 \underset{\sim}{k} \right)$$

$$= \left(v_2 w_3 - v_3 w_2 \right) \underset{\sim}{i} + \left(v_3 w_1 - v_1 w_3 \right) \underset{\sim}{j} + \left(v_1 w_2 - v_2 w_1 \right) \underset{\sim}{k}$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Products of $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$:

$\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ are mutually perpendicular unit vectors

$$\Rightarrow \quad \underset{\sim}{i} \cdot \underset{\sim}{i} = \underset{\sim}{j} \cdot \underset{\sim}{j} = \underset{\sim}{k} \cdot \underset{\sim}{k} = 1$$

$$\underset{\sim}{i} \cdot \underset{\sim}{j} = \underset{\sim}{i} \cdot \underset{\sim}{k} = \underset{\sim}{j} \cdot \underset{\sim}{k} = 0 .$$

$$\underset{\sim}{i} \times \underset{\sim}{i} = \underset{\sim}{j} \times \underset{\sim}{j} = \underset{\sim}{k} \times \underset{\sim}{k} = \underset{\sim}{0} .$$

What about $\underset{\sim}{i} \times \underset{\sim}{j}$ etc ?

$$\underset{\sim}{i} \times \underset{\sim}{j} = \underbrace{|\underset{\sim}{i}|}_{1}\,\underbrace{|\underset{\sim}{j}|}_{1}\,\underbrace{\sin\theta}_{1}\,\underbrace{\underset{\sim}{\hat{n}}}_{\text{vector } \perp \text{ to both } \underset{\sim}{i},\, \underset{\sim}{j}.}$$

$$\Rightarrow \quad \underset{\sim}{i} \times \underset{\sim}{j} = \pm \underset{\sim}{k} \quad \text{and likewise.}$$

It turns out

$$\underset{\sim}{i} \times \underset{\sim}{j} = +\underset{\sim}{k}, \qquad \underset{\sim}{j} \times \underset{\sim}{k} = +\underset{\sim}{i},$$

$$\underset{\sim}{k} \times \underset{\sim}{i} = +\underset{\sim}{j}$$

$$\underset{\sim}{j} \times \underset{\sim}{i} = -\underset{\sim}{k} \quad \text{etc.}$$

(To memorise : $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}, \underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k} \ldots$)

SCALAR TRIPLE PRODUCT

Given three vectors $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$,

the scalar triple product is

$$\underbrace{\underbrace{(\underset{\sim}{u} \times \underset{\sim}{v})}_{\text{Vector}} \cdot \underbrace{\underset{\sim}{w}}_{\text{Vector}}}_{\text{Scalar.}}$$

In components,

$$\begin{aligned} \underset{\sim}{u} \times \underset{\sim}{v} = & (u_2 v_3 - u_3 v_2)\, \underset{\sim}{i} \\ & + (u_3 v_1 - u_1 v_3)\, \underset{\sim}{j} \\ & + (u_1 v_2 - u_2 v_1)\, \underset{\sim}{k} \end{aligned}$$

$$\underset{\sim}{w} = w_1 \underset{\sim}{i} + w_2 \underset{\sim}{j} + w_3 \underset{\sim}{k}$$

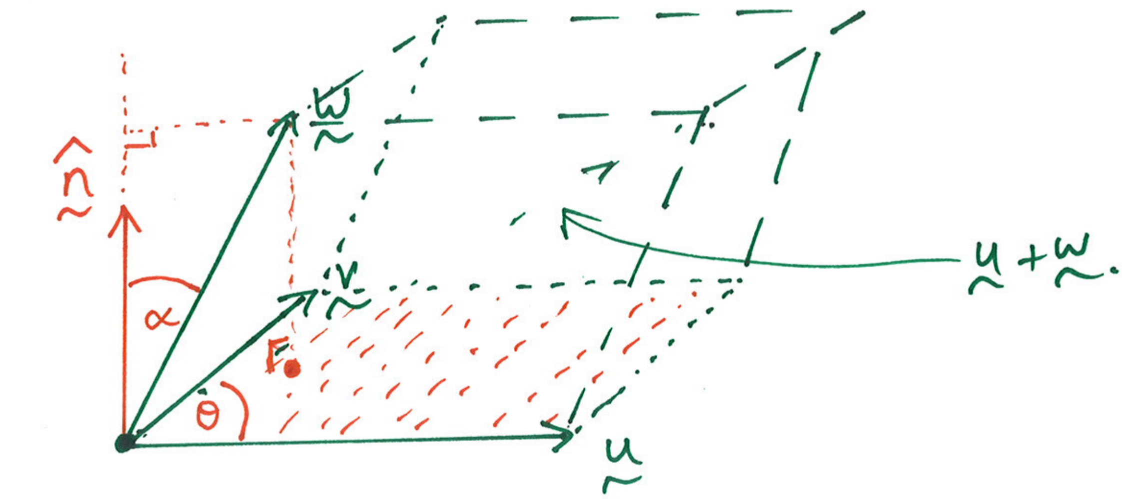
$$\begin{aligned} \Rightarrow (\underset{\sim}{u} \times \underset{\sim}{v}) \cdot \underset{\sim}{w} = & \; w_1 u_2 v_3 - w_1 u_3 v_2 \\ & + w_2 u_3 v_1 - w_2 u_1 v_3 \\ & + w_3 u_1 v_2 - w_3 u_2 v_1 \end{aligned}$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\text{So} \quad (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}$$
$$= (\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}$$
$$= -(\mathbf{v}\times\mathbf{u})\cdot\mathbf{w}$$

etc.

Geometrical picture :



$$\mathbf{u}\times\mathbf{v} = \underbrace{|u|\,|v|\,\sin\theta}_{\text{Area of `base'.}}\ \hat{\mathbf{n}}$$

$$\hat{\mathbf{n}}\cdot\mathbf{w} = |\hat{\mathbf{n}}|\,|\mathbf{w}|\cos\alpha$$
$$= 1\,|w|\cos\alpha$$
$$= \text{Height of } w \text{ above } uv \text{ plane.}$$

$$\Rightarrow (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \pm \text{ Volume of parallelepiped}$$

Depends if $\mathbf{u},\mathbf{v},\mathbf{w}$ are 'right handed' or 'left handed' triple

Example of sign-flip:

Suppose we reflect $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ in a mirror in the xy plane, to get $\underset{\sim}{u}', \underset{\sim}{v}', \underset{\sim}{w}'$

$$\rightarrow \quad \underset{\sim}{u}' = (u_1, u_2, -u_3) \qquad \text{etc.}$$

Then
$$(\underset{\sim}{u}' \times \underset{\sim}{v}') \cdot \underset{\sim}{w}' = \begin{vmatrix} u_1 & u_2 & -u_3 \\ v_1 & v_2 & -v_3 \\ w_1 & w_2 & -w_3 \end{vmatrix}$$

$$= - \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

VECTOR TRIPLE PRODUCT:

$$\underset{\sim}{u} \times (\underset{\sim}{v} \times \underset{\sim}{w}) = (\underset{\sim}{u} \cdot \underset{\sim}{w})\underset{\sim}{v} - (\underset{\sim}{u} \cdot \underset{\sim}{v})\underset{\sim}{w}$$

$$= -\underset{\sim}{u} \times (\underset{\sim}{w} \times \underset{\sim}{v})$$

$$= -(\underset{\sim}{v} \times \underset{\sim}{w}) \times \underset{\sim}{u} .$$

$$\neq (\underset{\sim}{u} \times \underset{\sim}{v}) \times \underset{\sim}{w}$$

Proof: write out components, Ex. 1.3.

PERMUTATIONS :

1, 2, 3

3, 1, 2

2, 3, 1

Even :

+

2, 1, 3

1, 3, 2

3, 2, 1.

Odd.

−

$$\underset{\sim}{u} \times (\underset{\sim}{v} \times \underset{\sim}{w}) = ?$$

$$\underset{\sim}{v} \times \underset{\sim}{w} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= (v_2 w_3 - v_3 w_2)\, \underset{\sim}{i} + \cdots$$

$$\underset{\sim}{u} \times (\underset{\sim}{v} \times \underset{\sim}{w}) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ u_1 & u_2 & u_3 \\ \begin{matrix} v_2 w_3 \\ - v_3 w_2 \end{matrix} & \cdots & \cdots \end{vmatrix}$$

1.8 GRADIENTS & DIRECTIONAL DERIVATIVES.

In Calculus II, you met functions of more than one variable e.g. $f(x,y)$ or $f(x,y,z)$.

Here f is a pure number $\rightarrow$ "scalar"

x, y, z is <u>position in space</u>

$\rightarrow$ $f(x,y,z) \equiv f(\underset{\sim}{r})$ is called a SCALAR FIELD.

Given $f(\underset{\sim}{r})$, we can define a vector field, the GRADIENT of f, by

$$\underset{\sim}{\nabla} f = \frac{\partial f}{\partial x}\underset{\sim}{i} + \frac{\partial f}{\partial y}\underset{\sim}{j} + \frac{\partial f}{\partial z}\underset{\sim}{k}$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$ as a row vector.

Suppose we start at $\underset{\sim}{r} = (x, y, z)$,

then move a <u>small</u> distance $d\underset{\sim}{r} = (dx, dy, dz)$

to $\underset{\sim}{r} + d\underset{\sim}{r} = (x+dx,\ y+dy,\ z+dz)$.

Consider what happens to f :

get a small change df,

$$df \equiv f(x+dx, y+dy, z+dz) - f(x,y,z) .$$

Now use Taylor series in 3 variables, and throw away 2nd and higher derivatives.

Result : $$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \cdot (dx, dy, dz)$$

$$df = (\underset{\sim}{\nabla f}) \cdot d\underset{\sim}{r}$$

Taylor series in 3 variables

You should remember the formula for a Taylor series in 1 variable, e.g.

$$f(x) = f(a) + \frac{df}{dx}(x-a) + \frac{1}{2!}\frac{d^2f}{dx^2}(x-a)^2 + \ldots$$

where the derivatives are evaluated at $x = a$.

Letting $x_0 = a, x = x_0 + \delta x$, we have the same thing as

$$f(x_0 + \delta x) = f(x_0) + \frac{df}{dx}(\delta x) + \frac{1}{2!}\frac{d^2f}{dx^2}(\delta x)^2 + \ldots$$

There is a similar version of Taylor's theorem if f takes more than one variable; specifically for 3 variables $f(x, y, z)$ we get

$$\begin{aligned} & f(x_0 + \delta x, y_0 + \delta y, z_0 + \delta z) \\ = \quad & f(x_0, y_0, z_0) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z \\ & \cancel{+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\delta x)^2} + \cancel{\frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\delta y)^2} + \cancel{\frac{1}{2}\frac{\partial^2 f}{\partial z^2}(\delta z)^2} \\ & \cancel{+\frac{\partial^2 f}{\partial x \partial y}(\delta x)(\delta y)} + \cancel{\frac{\partial^2 f}{\partial y \partial z}(\delta y)(\delta z)} + \cancel{\frac{\partial^2 f}{\partial x \partial z}(\delta x)(\delta z)} + \ldots \end{aligned}$$

Note that there are "mixed derivatives" at second and higher orders, like $\partial^2 f/\partial x \partial y$, etc.

But, if we take $\delta x, \delta y, \delta z$ all small enough that only first-order terms contribute , then we simply get

$$f(x_0 + \delta x, y_0 + \delta y, z_0 + \delta z) \approx f(x_0, y_0, z_0) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z$$

or , defining δf in the obvious way as

$$\begin{aligned} \delta f &= f(x_0 + \delta x, y_0 + \delta y, z_0 + \delta z) - f(x_0, y_0, z_0) \\ \delta f &\approx \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z \end{aligned}$$

Example 1·7 :

If $f(x,y,z) = x^2 \sin z$, calculate $\underset{\sim}{\nabla} f$

Take ~~Need~~ partial derivatives :

$$\frac{\partial f}{\partial x} = 2x \sin z$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} = x^2 \cos z$$

so $$\underset{\sim}{\nabla} f = (2x \sin z)\,\underset{\sim}{i} + 0\,\underset{\sim}{j} + (x^2 \cos z)\,\underset{\sim}{k} .$$

What does $\underset{\sim}{\nabla} f$ tell us ?

$\underset{\sim}{\nabla} f$ ~~the~~ depends on $\underset{\sim}{r}$.
(It is a vector field, see later).

$\underset{\sim}{\nabla} f$ at $\underset{\sim}{r}$ tells us how f is changing 'near' point $\underset{\sim}{r}$.

TANGENT PLANE & NORMAL LINE.

Last time we saw

$$\underset{\sim}{\nabla} f \equiv \frac{\partial f}{\partial x} \underset{\sim}{i} + \frac{\partial f}{\partial y} \underset{\sim}{j} + \frac{\partial f}{\partial z} \underset{\sim}{k}$$

and we showed

$$df = (\underset{\sim}{\nabla} f) \cdot d\underset{\sim}{r}$$

for any <u>small</u> displacement $d\underset{\sim}{r}$.

Now, consider a surface

$$f(\underset{\sim}{r}) = c \qquad \text{for constant } c.$$

Let $\underset{\sim}{r}_1$ be any point on the surface,

so $f(\underset{\sim}{r}_1) = c$

Let $d\underset{\sim}{r}$ be a small displacement <u>on</u> the surface, so

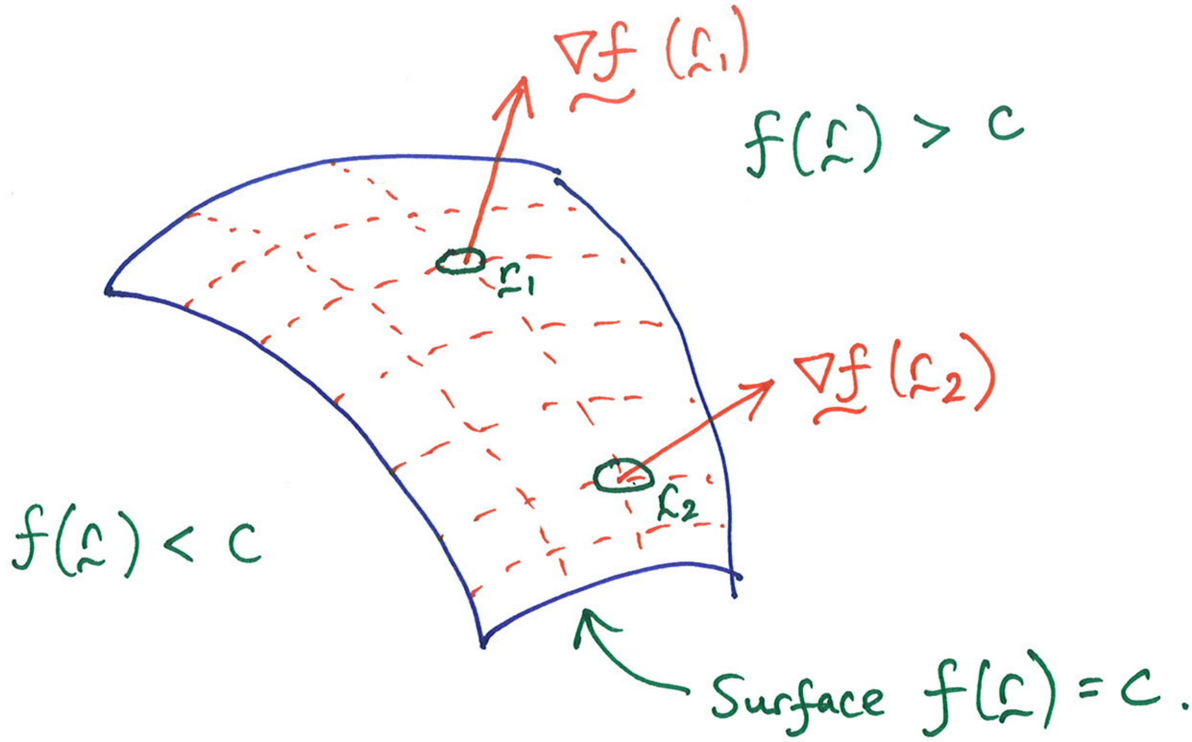
$$f(\underset{\sim}{r}_1 + d\underset{\sim}{r}) = c \qquad \text{also.}$$

Now $df = f(\underset{\sim}{r}_1 + d\underset{\sim}{r}) - f(\underset{\sim}{r}_1) = 0$

and $df = (\underset{\sim}{\nabla} f) \cdot d\underset{\sim}{r}$

so $(\underset{\sim}{\nabla} f) \cdot d\underset{\sim}{r} = 0$

$\Rightarrow$ $\underset{\sim}{\nabla} f$ and $d\underset{\sim}{r}$ are perpendicular.



$\underset{\sim}{\nabla f}$ is perpendicular to <u>local</u> surface $f(\underset{\sim}{r}) = \text{const}$

Tangent plane:

Suppose we are given a surface $f(\underset{\sim}{r}) = c$ and a point $\underset{\sim}{p}$ on the surface.

$\underset{\sim}{\nabla f}\Big|_{\underset{\sim}{p}}$ is normal to surface, at $\underset{\sim}{p}$

$\Rightarrow$ tangent plane through $\underset{\sim}{p}$ is $(\underset{\sim}{r} - \underset{\sim}{p}) \cdot \underset{\sim}{\nabla f}\Big|_{\underset{\sim}{p}} = 0$.

Normal line through $\underset{\sim}{p}$

is $\underset{\sim}{r} = \underset{\sim}{p} + t \, \underset{\sim}{\nabla f}\Big|_{\underset{\sim}{p}}$

Parameter.

Exercise 1.5

Find equations for the tangent plane & normal line to the surface

$$x^2 + 3yz + 4xy = 27$$

at the point $P_0 = (3, 1, 2)$:

$$f(\underset{\sim}{r}) = x^2 + 3yz + 4xy$$

any $\underset{\sim}{r}$ $\rightarrow$ $$\underset{\sim}{\nabla} f = (2x + 4y)\,\underset{\sim}{i} + (3z + 4x)\,\underset{\sim}{j} + (3y)\,\underset{\sim}{k}\,.$$

at P_0 : $$\underset{\sim}{\nabla} f\Big|_{P_0} = 10\,\underset{\sim}{i} + 18\,\underset{\sim}{j} + 3\,\underset{\sim}{k}$$

Tan. plane $(\underset{\sim}{r} - \underset{\sim}{P_0}) \cdot (\underset{\sim}{\nabla} f|_{P_0}) = 0$

$$(x-3,\ y-1,\ z-2) \cdot (10,\ 18,\ \overset{3}{\cancel{3}}) = 0$$

$$10(x-3) + 18(y-1) + 3(z-2) = 0$$

$$10x + 18y + 3z = 54\,.$$

Normal line : $\underset{\sim}{r} = \underset{\sim}{P_0} + t\,(\cancel{\nabla f}\ \underset{\sim}{\nabla} f|_{P_0})$

$$= (3, 1, 2) + t\,(10, 18, 3)$$

$$= (3 + 10t)\,\underset{\sim}{i} + (1 + 18t)\,\underset{\sim}{j} + (2 + 3t)\,\underset{\sim}{k}\,.$$

$$x = 3 + 10t$$

$$y = 1 + 18t$$

$$z = 2 + 3t$$

$$t = \frac{x-3}{10}.$$

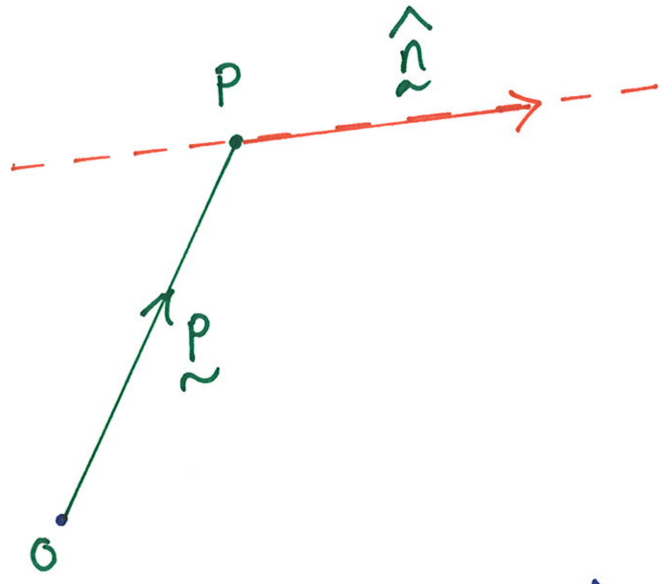
$$y = 1 + 18\left(\frac{x-3}{10}\right)$$

$$z = 2 + 3\left(\frac{x-3}{10}\right).$$

DIRECTIONAL DERIVATIVE :

Suppose we want to calculate the rate of change of $f(\underset{\sim}{r})$ in a specific direction.

A direction is specified by a unit vector $\hat{\underset{\sim}{n}}$.



Line through P in direction $\hat{\underset{\sim}{n}}$ is

$$\underset{\sim}{r} = \underset{\sim}{p} + s\,\hat{\underset{\sim}{n}}, \qquad -\infty < s < \infty.$$

Distance from P along line.

Consider a small change ds

Get $\quad d\underset{\sim}{r} = ds\,\hat{\underset{\sim}{n}}$

$$df = (\underset{\sim}{\nabla} f) . d\underset{\sim}{r}$$

$$= (\underset{\sim}{\nabla} f) . \hat{\underset{\sim}{n}} \; ds$$

so $$\frac{df}{ds} = \underbrace{(\underset{\sim}{\nabla f}) \cdot \underset{\sim}{\hat{n}}}_{\text{Directional derivative of } f \text{ along } \underset{\sim}{\hat{n}}.}$$

$$= |\underset{\sim}{\nabla f}| \cdot 1 \cdot \cos\theta$$

$$= |\underset{\sim}{\nabla f}| \cos\theta \, .$$

At fixed point $\underset{\sim}{r}$. (θ is angle between $\underset{\sim}{\nabla f}$ and $\underset{\sim}{\hat{n}}$)

$\frac{df}{ds}$ is Max. when $\cos\theta = +1$, $\Rightarrow \underset{\sim}{\hat{n}} \parallel \underset{\sim}{\nabla f}$.

$\frac{df}{ds}$ is min. when $\cos\theta = -1$, $\underset{\sim}{\hat{n}} \parallel -\underset{\sim}{\nabla f}$.

$\frac{df}{ds}$ is zero when $\cos\theta = 0$, $\underset{\sim}{\hat{n}} \perp \underset{\sim}{\nabla f}$.

(so $\underset{\sim}{\hat{n}}$ lies in tangent plane to $f(\underset{\sim}{r}) = c$)

Example 1·8 :

Find the directions in which

$$f(x, y, z) = \frac{x}{y} - yz$$

increases and decreases most rapidly at the point $P = (4, 1, 1)$.

At Any $\underset{\sim}{r}$: $\underset{\sim}{\nabla} f = \frac{1}{y}\underset{\sim}{i} + \left(\frac{-x}{y^2} - z\right)\underset{\sim}{j} - y\underset{\sim}{k}$.

At P : $\underset{\sim}{\nabla} f = 1\underset{\sim}{i} - 5\underset{\sim}{j} - \underset{\sim}{k}$.

$$|\underset{\sim}{\nabla} f| = \sqrt{27}$$

Unit vector $\underset{\sim}{\hat{n}}$ // to $\underset{\sim}{\nabla} f$ is

$$\underset{\sim}{\hat{n}} = \frac{1}{\sqrt{27}}(1, -5, -1)$$

so f increases most rapidly along $\underset{\sim}{\hat{n}}$ $\left(\frac{df}{ds} = \sqrt{27}\right)$ and decreases most rapidly along $-\underset{\sim}{\hat{n}}$, $\frac{df}{ds} = -\sqrt{27}$.