Queen Mary
University of London

# B. Sc. Examination by course unit 2010

## MTH5102 Calculus III

**Duration: 2 hours**

**Date and time: 20th May 2010, 2:30pm**

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**Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.**

**You should attempt all questions. Marks awarded are shown next to the questions.**

**Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.**

**Complete all rough workings in the answer book and cross through any work which is not to be assessed.**

**Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.**

**Exam papers must not be removed from the examination room.**

**Examiner(s): W.J. Sutherland**

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**Question 1** [10]

(a) Find the gradient of $V = x^2 + y^2 - 2z$

(b) Describe the surface $V = 11$.

(c) Find an equation for the plane tangent to this surface at the point $\mathbf{P} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

**Question 2** [11]

(a) The parametric curve $C$ is given by

$$\mathbf{r} = a\cos t\ \mathbf{i} + a\sin t\ \mathbf{j} + bt\ \mathbf{k}$$

where $a, b$ are constants and $t$ is the parameter. Describe the curve. Evaluate the arc-length of the curve between the points $(a, 0, 0)$ and $(a, 0, 4\pi b)$.

(b) Another curve is given in plane polar coordinates by $r = c(1 + \cos\theta)$, where $c$ is a constant. Sketch this curve, and evaluate the area enclosed inside it.

**Question 3** [10]

Evaluate the line integral $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = (x-2z^2)\mathbf{i}+(y+z)\mathbf{j}+xz\mathbf{k}$ and $\mathcal{C}$ is the curve from (0, 0, 0) to (3, 6, 9) described in parametric form as $\mathcal{C} : \mathbf{r} = t\mathbf{i}+2t\mathbf{j}+t^2\mathbf{k}$, where the parameter $t$ has the range $0 \le t \le 3$.

**Question 4** [11]

(a) Given a scalar field $U$ and a vector field $\mathbf{F}$, write down the expressions for $\nabla U$ and $\nabla \cdot \mathbf{F}$ in Cartesian coordinates.

(b) Prove that $\nabla \cdot (U\mathbf{F}) = U\nabla \cdot \mathbf{F} + (\nabla U) \cdot \mathbf{F}$.

(c) Hence prove that for two scalar fields $U, V$, the Laplacian of the product $UV$ is

$$\nabla^2(UV) = U\nabla^2 V + V\nabla^2 U + 2(\nabla U) \cdot (\nabla V)$$

**Question 5** [10]

(a) State the Divergence Theorem.

(b) The vector field $\mathbf{F}$ is given by $\mathbf{F} = (2x + y)\mathbf{i} + (y + xz)\mathbf{j} + x^2\mathbf{k}$. Evaluate the divergence $\nabla \cdot \mathbf{F}$, and thus using the Divergence Theorem, evaluate the surface integral $\int_S \mathbf{F}.d\mathbf{S}$ summed over all faces of the cuboid $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, where $d\mathbf{S}$ is taken in the outward normal direction.

**Question 6** [12]

For each of the following vector fields $\mathbf{F}$, calculate the curl $\nabla \times \mathbf{F}$. If there is a scalar field $\Phi$ such that $\mathbf{F} = \nabla\Phi$, find the most general such $\Phi$; otherwise give a reason why no such $\Phi$ exists.

(a) $\mathbf{F} = x^2\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ ,

(b) $\mathbf{F} = (3x^2 + z)\mathbf{i} + 2y\mathbf{j} + x\mathbf{k}$ .

For the field in (b) above, explain why $\int_C \mathbf{F}.d\mathbf{r}$ around any closed curve $C$ is zero.

**Question 7** [12]

In spherical polar coordinates $(r, \theta, \phi)$, the position vector is

$$\mathbf{r} = r\sin\theta\,\cos\phi\,\mathbf{i} + r\sin\theta\,\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k}$$

(a) Calculate $\partial\mathbf{r}/\partial\theta$ and $\partial\mathbf{r}/\partial\phi$, and show they are orthogonal.

(b) Hence prove that the area element $d\mathbf{S}$ on a surface of constant $r$ is

$$d\mathbf{S} = r^2 \sin\theta\,\mathbf{e}_r\,d\theta\,d\phi$$

where $\mathbf{e}_r$ is the unit vector parallel to $\mathbf{r}$.

(c) The vector field $\mathbf{F}$ is given in spherical polar coordinates by $\mathbf{F} = 2\cos\theta\,\mathbf{e}_r + \sin\theta\,\mathbf{e}_\theta$. The surface $S$ is the hemisphere with $r = a$, $z \geq 0$, for constant $a$. Using results above, evaluate the surface integral

$$\int_S \mathbf{F}.d\mathbf{S}$$

where the normal is taken in the direction away from the origin.

 

**Question 8** [12]

(a) State Parseval's theorem, without proof.

(b) Show that the function $f(x)$ with period $2\pi$ whose values in $-\pi \le x \le \pi$ are given by $f(x) = x^2$ has the Fourier series

$$S(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

Explain briefly why there are no $\sin nx$ terms in the series.

(c) By evaluating $S(x)$ at a suitable $x$, prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

**Question 9** [12]

(a) For a scalar field $U$ defined in cylindrical polar coordinates $(\rho, \phi, z)$, show that the Laplacian $\nabla^2 U$ is given by

$$\nabla^2 U = \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

(You may quote results from the Appendix).

(b) The general $z$-independent solution of Laplace's equation in cylindrical polars is a sum of terms of the form

$$\begin{aligned} U(\rho,\phi) &= (A_0\phi + B_0)(C_0 \ln \rho + D_0) \\ &\quad + \sum_{m=1}^{\infty} (A_m \cos m\phi + B_m \sin m\phi)\left(C_m \rho^m + D_m \rho^{-m}\right) \end{aligned}$$

where $m$ is any positive integer, and the $A_i, B_i, C_i, D_i$ are arbitrary constants.

Find the specific solution $U(\rho, \phi)$ of Laplace's equation in the disk $\rho \le 2$, with boundary condition given by $U = 1 + \cos^2 \phi$ on the circle $\rho = 2$, and $U$ bounded at the origin.

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**End of Paper**

**( An Appendix of one page follows )**

## Appendix

You are reminded of the following, which you may use without proof:

In orthogonal curvilinear coordinates $(u_1,\, u_2,\, u_3)$, with corresponding unit vectors $\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3$ and arc-length parameters $h_1,\, h_2,\, h_3$, the gradient of a scalar field $f$ is given by

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\mathbf{e}_3.$$

The divergence of a vector field $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ is given by

$$\nabla.\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3F_1) \;+\; \frac{\partial}{\partial u_2}(h_3h_1F_2) \;+\; \frac{\partial}{\partial u_3}(h_1h_2F_3)\right] ,$$

and the curl of the same vector field is given by

$$\nabla\times\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} .$$

In spherical polar coordinates $(u_1,\, u_2,\, u_3) \equiv (r,\, \theta,\, \phi)$, the arc-length parameters are $h_1 = 1,\, h_2 = r,\, h_3 = r\sin\theta$.
In cylindrical polar coordinates $(u_1,\, u_2,\, u_3) \equiv (\rho,\, \phi,\, z)$, the arc-length parameters are
$h_1 = 1,\, h_2 = \rho,\, h_3 = 1$.