MTH 5102 Calculus III - MAY 2010.

Q1: (a) $V = x^2 + y^2 - 2z$

$$\nabla V = 2x\underset{\sim}{i} + 2y\underset{\sim}{j} - 2\underset{\sim}{k}. \qquad [3].$$

(No marks if answer is scalar.)

(b) $V = 11$:
a paraboloid of revolution, axis the z-axis,
[3 : 1 mark each].

(c) $\underset{\sim}{p} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$.

Tangent plane $(\underset{\sim}{r} - \underset{\sim}{p}) \cdot (\nabla V)_p = 0$

or $\underset{\sim}{r} \cdot (\nabla V)_{\underset{\sim}{p}} = \underset{\sim}{p} \cdot (\nabla V)_p$. [1]

$$(\nabla V)_{\underset{\sim}{p}} = 6\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}. \qquad [1]$$

$\Rightarrow 6x + 4y - 2z = 18 + 8 - 2$
$6x + 4y - 2z = 24$ [2 :
or $3x + 2y - z = 12$. 1 per side].

Notes : (a) straightforward
(b) bookwork
(c) straightforward

(Main sit).

Q2 (a) C is a helix, axis the z-axis, of radius a. [3].

End-points $\Rightarrow$ $t = 0$, $t = 4\pi$

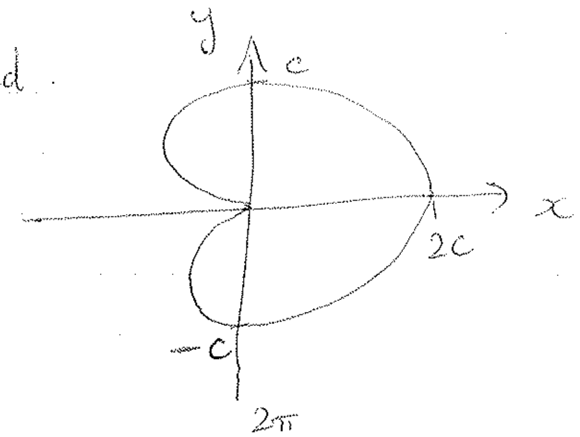
Arc-length $s = \displaystyle\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$. [1]

$$= \int_0^{4\pi} \sqrt{[a^2 \sin^2 t + a^2 \cos^2 t + b^2]}\, dt$$

$$= \int_0^{4\pi} \sqrt{a^2 + b^2}\, dt$$

$$= 4\pi \sqrt{a^2 + b^2}.$$ [2].

(b) Cardioid.



$$\text{Area} = \frac{1}{2} \int_0^{2\pi} r^2(\theta)\, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} c^2 (1 + \cos\theta)^2\, d\theta$$

$$= \frac{1}{2} c^2 \int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta\, d\theta$$

$$= \frac{1}{2} c^2 [2\pi + 0 + \pi] = \frac{3\pi}{2} c^2.$$ [4]

Notes: (a), (b) similar to coursework.

Q3: $\underset{\sim}{F} = (x - 2z^2)\underset{\sim}{i} + (y+z)\underset{\sim}{j} + xz\underset{\sim}{k}$.

Curve C: $\underset{\sim}{r} = t\underset{\sim}{i} + 2t\underset{\sim}{j} + t^2\underset{\sim}{k}$.

$$F(\underset{\sim}{r}(t)) = (t - 2t^4)\underset{\sim}{i} + (2t + t^2)\underset{\sim}{j} + t^3\underset{\sim}{k} \qquad [2]$$

$$\frac{d\underset{\sim}{r}}{dt} = 1\underset{\sim}{i} + 2\underset{\sim}{j} + 2t\underset{\sim}{k}. \qquad [2]$$

$$\underset{\sim}{F} \cdot d\underset{\sim}{r} = (t - 2t^4) + 2(2t + t^2) + 2t^4$$

$$= 5t + 2t^2. \qquad [2]$$

$$\int \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^3 5t + 2t^2 \, dt \qquad [2]$$

$$= \left[\frac{5}{2}t^2 + \frac{2}{3}t^3 \right]_0^3$$

$$= \frac{45}{2} + 18 = \frac{81}{2}. \qquad [2].$$

Notes: all similar to coursework.

Q4 (a) $$\underset{\sim}{\nabla} U = \frac{\partial U}{\partial x}\underset{\sim}{i} + \frac{\partial U}{\partial y}\underset{\sim}{j} + \frac{\partial U}{\partial z}\underset{\sim}{k} \qquad [1]$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \qquad [1]$$

(b) $$\underset{\sim}{\nabla}\cdot(U\underset{\sim}{F}) = \frac{\partial}{\partial x}(UF_1) + \frac{\partial}{\partial y}(UF_2) + \frac{\partial}{\partial z}(UF_3)$$

$$= \frac{\partial U}{\partial x}F_1 + U\frac{\partial F_1}{\partial x} + \dots$$

$$= U\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$$

$$+ \frac{\partial U}{\partial x}F_1 + \frac{\partial U}{\partial y}F_2 + \frac{\partial U}{\partial z}F_3.$$

$$= U(\underset{\sim}{\nabla}\cdot\underset{\sim}{F}) + (\underset{\sim}{\nabla}U)\cdot\underset{\sim}{F}. \qquad [4]$$

(c) $$\nabla^2(UV) = \underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}(UV))$$

$$= \underset{\sim}{\nabla}\cdot(U\underset{\sim}{\nabla}V + V\underset{\sim}{\nabla}U)$$

Use (b) with $\underset{\sim}{F} = \underset{\sim}{\nabla}V$, $\underset{\sim}{\nabla}U$ respectively:

$$= \underset{\sim}{\nabla}\cdot(U\underset{\sim}{\nabla}V) + \underset{\sim}{\nabla}\cdot(V\underset{\sim}{\nabla}U)$$

$$= U\underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}V) + (\underset{\sim}{\nabla}U)\cdot(\underset{\sim}{\nabla}V)$$

$$+ V\underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}U) + (\underset{\sim}{\nabla}V)\cdot(\underset{\sim}{\nabla}U)$$

$$= U\nabla^2V + V\nabla^2U + 2(\underset{\sim}{\nabla}V)\cdot(\underset{\sim}{\nabla}U).$$

[5].

(a), (b) bookwork

(c) new.

Q5 (a) : $\int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{F} \, dV = \int_S \underset{\sim}{F} \cdot d\underset{\sim}{S}$ [2]

where $\underset{\sim}{F}$ is suitably differentiable, S is a closed ~~simply connected~~ orientable surface bounding volume V, and $d\underset{\sim}{S}$ ~~is~~ $= \underset{\sim}{n}\, dS$ is in the outward direction. [3]

(b) $\underset{\sim}{F} = (2x+y)\underset{\sim}{i} + (y+xz)\underset{\sim}{j} + x^2\underset{\sim}{k}$.

$\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 2+1+0 = 3$. [2]

$\int_S \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_V (\underset{\sim}{\nabla} \cdot \underset{\sim}{F}) \, dV$

$= \iiint 3 \, dV = 3abc$. [3].

~~Q6 (a)~~

(a) bookwork.

(b) similar to coursework.

[Similar to ~~x~~ courseworks]

Q6 (a) $\underset{\sim}{F} = x^2\underset{\sim}{i} + yz\underset{\sim}{j} + xz\underset{\sim}{k}$.

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & yz & xz \end{vmatrix}$$

$$= \underset{\sim}{i}(0-y) + \underset{\sim}{j}(0-z) + \underset{\sim}{k}(0-0)$$

$$= -y\underset{\sim}{i} - z\underset{\sim}{j}. \qquad [2]$$

$\underset{\sim}{\nabla} \times \underset{\sim}{F} \neq 0$, $\quad \underset{\sim}{\nabla} \times (\nabla\Phi) \equiv 0$ for any Φ

Therefore no Φ is possible. [2].

(b) $\underset{\sim}{F} = (3x^2+z)\underset{\sim}{i} + 2y\underset{\sim}{j} + x\underset{\sim}{k}$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2+z & 2y & x \end{vmatrix}$$

$$= \underset{\sim}{i}(0-0) + \underset{\sim}{j}(1-1) + \underset{\sim}{k}(0-0)$$

$$= \underset{\sim}{0}. \qquad [2]$$

$$\Phi = x^3 + xz + y^2 + c, \quad c = \text{const}.$$

(Several routes possible) [3].

Stokes' thm: $\displaystyle\int_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_S (\underset{\sim}{\nabla}\times\underset{\sim}{F})\cdot d\underset{\sim}{S}$ for closed curve C bounding surface S.

$= 0$.

[3].

[(a), (b) bookwork
(c) unseen.]

Q7: (a) $\underset{\sim}{r} = r\sin\theta\cos\phi\,\underset{\sim}{i} + r\sin\theta\sin\phi\,\underset{\sim}{j} + r\cos\theta\,\underset{\sim}{k}$.

$$\frac{\partial \underset{\sim}{r}}{\partial\theta} = r\cos\theta\cos\phi\,\underset{\sim}{i} + r\cos\theta\sin\phi\,\underset{\sim}{j} - r\sin\theta\,\underset{\sim}{k}. \quad [1]$$

$$\frac{\partial \underset{\sim}{r}}{\partial\phi} = -r\sin\theta\sin\phi\,\underset{\sim}{i} + r\sin\theta\cos\phi\,\underset{\sim}{j} + 0\,\underset{\sim}{k}. \quad [1]$$

$$\frac{\partial \underset{\sim}{r}}{\partial\theta}\cdot\frac{\partial \underset{\sim}{r}}{\partial\phi} = -r^2\cos\theta\cos\phi\sin\theta\sin\phi + r^2\cos\theta\sin\phi\sin\theta\cos\phi = 0 \quad [1].$$

(b) $$d\underset{\sim}{S} = \left(\frac{\partial \underset{\sim}{r}}{\partial\theta}\right)\times\left(\frac{\partial \underset{\sim}{r}}{\partial\phi}\right)d\theta\,d\phi. \quad [2]$$

$$= r^2\begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix}\times\begin{pmatrix}-\sin\theta\sin\phi\\ \sin\theta\cos\phi\\ 0\end{pmatrix}d\theta\,d\phi$$

$$= r^2\begin{pmatrix}0+\sin^2\theta\cos\phi\\ \sin^2\theta\sin\phi+0\\ \sin\theta\cos\theta\cos^2\phi+\sin\theta\cos\theta\sin^2\phi\end{pmatrix}d\theta\,d\phi$$

$$= r^2\sin\theta\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}d\theta\,d\phi. \quad [1]$$

$$= r^2\sin\theta\,\underset{\sim}{e}_r\,d\theta\,d\phi \quad [1].$$

Q7 (c) $\underset{\sim}{F} = 2\cos\theta \, \underset{\sim}{e}_r + \sin\theta \, \underset{\sim}{e}_{\cancel{\phi}\theta}$.

$$\underset{\sim}{F} \cdot d\underset{\sim}{S} = 2\cos\theta \cdot r^2 \sin\theta \; d\theta \; d\phi . \qquad [21]$$

Hemisphere : $r = a, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le 2\pi$.

$$\int_S \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_0^{2\pi}\int_0^{\pi/2} 2a^2 \sin\theta\cos\theta \; d\theta \; d\phi . \qquad [2]$$

$$= a^2 . (2\pi) \int_0^{\pi/2} \sin 2\theta \; d\theta .$$

$$= a^2 . \, 2\pi \left[-\frac{1}{2}\cos 2\theta \right]_0^{\pi/2}$$

$$= a^2 . \, 2\pi \left[\frac{1}{2} + \frac{1}{2} \right]$$

$$= 2\pi a^2 . \qquad [2] .$$

Q 8 (a) If $f(x)$ has a Fourier series defined by

$$S(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

then

$$\int_{-\pi}^{\pi} f(x)^2 \, dx = \frac{1}{2}\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

[3]

(b) In standard form,

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos mx \, dx \qquad [1]$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx \, dx. \qquad [1]$$

For $f(x) = x^2$, f is even, $\sin mx$ is odd,

$$\Rightarrow b_m = 0 \quad \forall m. \qquad [1]$$

For $m > 0$:

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos mx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos mx \, dx$$

$$= \frac{2}{\pi}\left\{ \left[x^2 \cdot \frac{1}{m}\sin mx\right]_0^{\pi} - \int_0^{\pi} 2x \cdot \frac{1}{m}\sin mx \, dx \right\}$$

$$= \frac{2}{\pi}\left\{ 0 - \left[2x \cdot \frac{(-1)}{m^2}\cos mx\right]_0^{\pi} + \int_0^{\pi} 2 \cdot \frac{-1}{m^2}\cos mx \, dx \right\}$$

$$= \frac{2}{\pi}\left\{ 0 + \frac{1}{m^2}(2\pi\cos(m\pi)) - \left[\frac{2}{m^3}\sin mx\right]_0^{\pi} \right\}$$

$$= \frac{4}{m^2}\cos(m\pi) = \frac{4}{m^2}(-1)^m.$$

Q8 : ctd.

$$m = 0 : \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$
$$= \frac{1}{\pi}\left[\frac{1}{3}x^3\right]_{-\pi}^{\pi}$$
$$= \frac{2}{3}\pi^2 .$$

$$\cancel{S(x) = \frac{1}{2}a_0 + \sum_m a_m \cos mx + \sum}$$

$$S(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$
$$= \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx . \qquad [3]$$

(c) Use $x = \pi$, $f(x) = \pi^2$
(and $x = -\pi$, $f(x) = \pi^2$ so continuous,

$$\Rightarrow S(\pi) = \pi^2 .$$

$$\pi^2 = \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi$$
$$= \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n$$
$$= \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{+1}{n^2}$$
$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} . \qquad [3].$$

[(a) bookwork (b) unseen
(c) similar to coursework.

Q9 (a) From Appendix:

Cylindrical polars: $h_1 = h_\rho = 1$,

$h_2 = h_\phi = \rho$

$h_3 = h_z = 1$.

$$\Rightarrow \quad \nabla U = \left(\frac{\partial U}{\partial \rho}\right)\underset{\sim}{e}_\rho + \left(\frac{1}{\rho}\frac{\partial U}{\partial \phi}\right)\underset{\sim}{e}_\phi + \left(\frac{\partial U}{\partial z}\right)\underset{\sim}{e}_z .$$

[3]

$$\nabla^2 U = \underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla} U) = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(1\cdot\left(\frac{1}{\rho}\frac{\partial U}{\partial \phi}\right)\right) + \frac{\partial}{\partial z}\left(1\rho\frac{\partial U}{\partial z}\right)\right]$$

$$= \frac{1}{\rho}\left(1\frac{\partial U}{\partial \rho} + \rho\frac{\partial^2 U}{\partial \rho^2}\right) + \frac{1}{\rho}\cdot\frac{1}{\rho}\frac{\partial^2 U}{\partial \phi^2} + \frac{1}{\rho}\cdot\rho\frac{\partial^2 U}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} .$$

[4].

(b) Use given general form.

On $\rho = 2$, $U = 1 + \cos^2\phi$

$= 1 + \frac{1}{2}(1 + \cos 2\phi)$

Bounded at $\rho = 0 \Rightarrow D_m = 0 \ \forall (m > 0)$

$C_0 = 0$.

Single-valuedness $\Rightarrow A_0 = 0$.

Q9 continued

$$\Rightarrow \quad U(\rho,\phi) = B_0 D_0 + \sum_{m=1}^{\infty} (A_m \cos m\phi + B_m \sin m\phi) C_m \rho^m .$$

$$\text{w.l.o.g} \quad = b_0 + \sum_{m=1}^{\infty} (a_m \cos m\phi + b_m \sin m\phi)\rho^m .$$

Only $m = 0$ and $m = 2$ in boundary condition, so ~~lets~~ try

$$U(\rho,\phi) = b_0 + a_2 (\cos 2\phi)\rho^2$$

and all others zero.

Given $U(2,\phi) = \frac{3}{2} + \frac{1}{2}\cos 2\phi$

$$\Rightarrow \quad b_0 = \frac{3}{2}, \quad a_2 = \frac{1}{8}$$

$$U(\rho,\phi) = \frac{3}{2} + \frac{1}{8}\rho^2 \cos 2\phi . \quad [5]$$

[(a) bookwork
(b) similar to coursework.]