Queen Mary
University of London

B. Sc. Examination by course unit 2009

MTH5102 Calculus III

Duration: 2 hours

Date and time: 27th May 2009, 10 AM

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators may be used in this examination, but any programming, graph plotting or algebraic facility may not be used. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): M.A.H. MacCallum

Question 1 [12]

(a) Find the gradient of $V = 9z^2 - x^2 - 4y^2$.

(b) Sketch the surface $V = 5$ and describe its shape.

(c) Find an equation for the plane tangent to this surface at $\mathbf{P} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Question 2 [11]

Evaluate the integral $\int_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ where $\mathbf{F} = (2x - z^4)\mathbf{i} + (y + 2z)\mathbf{j} - xy\mathbf{k}$ and $\mathcal{C}$ is the path going from (0, 0, 0) to (1, 1, 1) described in parametric form as $\mathcal{C} : \mathbf{r} = t^2\mathbf{i} + t^3\mathbf{j} + t\mathbf{k}$, where the parameter t obeys $0 \le t \le 1$.

Question 3 [10]

Show that $\mathbf{F} = (x + y^2)\mathbf{i} + (3y + xz)\mathbf{j} + (2z + y^3)\mathbf{k}$ has a constant divergence, and hence, using the Divergence Theorem, evaluate $\int_S \mathbf{F}.\mathrm{d}\mathbf{S}$ over the surface of the sphere of radius a centred at (1, 0, −1), where $\mathbf{S}$ is taken in the direction of the outward normal on the sphere.

Question 4 A surface S has the equation [8]

$$z = x(1 - x)y(1 - y)$$

for $0 \le x \le 1$ and $0 \le y \le 1$. The unit square R is $0 \le x \le 1$, $0 \le y \le 1$. Describe the boundaries of S and R. Hence, explain carefully why it is that for any (smooth) vector field $\mathbf{F}$ the integral $\int_S \nabla \times \mathbf{F}.\mathrm{d}\mathbf{S}$ is the same as $\int_R \nabla \times \mathbf{F}.\mathrm{d}\mathbf{S}$, assuming the normal used for both surfaces has a positive $\mathbf{k}$ component.

Question 5 [12]

For each of the following vector fields $\mathbf{F}$, calculate the curl $\nabla \times \mathbf{F}$. If there is a Φ such that $\mathbf{F} = \nabla\Phi$, find the most general such Φ.

(a) $\mathbf{F} = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$,

(b) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$.

Question 6 [13]

Calculate $\nabla\Phi$ in spherical polar coordinates, where $\Phi = r^2 \sin^2\theta \cos(2\phi)$. Show that this Φ satisfies Laplace's equation $\nabla \cdot (\nabla\Phi) = \nabla^2\Phi = 0$.

Question 7 The function $u(x)$ obeys the differential equation [11]

$$x^2(u'' + u') + (x-2)u = 0,$$

where the prime denotes the derivative with respect to x, i.e. $u' = \dfrac{\mathrm{d}u}{\mathrm{d}x}$.

Find the nature of the point $x = 0$ (ordinary, regular singular or irregular singular).

Using Frobenius's method, find the solution for the smaller root of the indicial equation.

[You may quote the following results. If $xf = f_0 + f_1 x + f_2 x^2 + \ldots$ and $x^2 g = g_0 + g_1 x + g_2 x^2 + \ldots$ where f and g are the functions appearing in the standard form

$$u'' + f(x)u' + g(x)u = 0,$$

of the equation, then the indicial equation is

$$c(c-1) + f_0 c + g_0 = 0$$

and the recurrence relation is

$$\begin{aligned} a_r\{(r+c)(r+c-1) + f_0(r+c) + g_0\} + a_{r-1}\{(r+c-1)f_1 + g_1\} & \\ +a_{r-2}\{(r+c-2)f_2 + g_2\} + \ldots + a_0\{cf_r + g_r\} &= 0\ .] \end{aligned}$$

Question 8 [10]

Show that the even function with period 2π whose values in $[0, \pi]$ are given by $f(x) = e^x$ has the Fourier series

$$S(x) = \frac{e^\pi - 1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n e^\pi}{1 + n^2} \cos nx.$$

[You may quote the result that

$$\int_0^\pi e^x \cos(nx)\mathrm{d}x = \left[\frac{e^x \cos(nx)}{n^2}\right]_0^\pi - \frac{1}{n^2}\int_0^\pi e^x \cos(nx)\mathrm{d}x$$

which comes from integration by parts.]

Question 9 [13]

The function $V(x, y)$ satisfies $\nabla^2 V = 0$ in the region $0 < x < a,\ 0 < y < b$ and the boundary conditions that $V = g(y)$ on $x = a$, for some function $g(y)$, and $V = 0$ on the other three sides of the region (and at all the corners).

(a) Show that the solution for V takes the form of a series

$$V = \sum_{n=1}^{\infty} K_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}.$$

[You may assume that the general solution of Laplace's equation in the rectangle takes the form of a sum of terms of the forms

$$(A_0 + B_0 x)(C_0 + D_0 y),$$
$$\left(A_n \cos \frac{n\pi x}{a} + B_n \sin \frac{n\pi x}{a}\right)\left(C_n \cosh \frac{n\pi y}{a} + D_n \sinh \frac{n\pi y}{a}\right),$$
$$\left(a_n \cosh \frac{n\pi x}{b} + b_n \sinh \frac{n\pi x}{b}\right)\left(c_n \cos \frac{n\pi y}{b} + d_n \sin \frac{n\pi y}{b}\right),$$

where n is any positive integer, and the a_i, b_i, c_i, d_i, A_i, B_i, C_i and D_i are constants.]

(b) For the specific case where $g(y) = V_0$, where V_0 is a constant, find the coefficients K_n and hence find V.

[You may assume that the odd function $f(y)$ with period $2b$ and value V_0 in $(0, \pi)$ has the Fourier series

$$\sum_{k=1}^{\infty} \frac{4V_0}{(2k+1)\pi} \sin(2k+1)\frac{\pi y}{b}\ .]$$

End of Paper: an Appendix of 1 page follows

Appendix

You are reminded of the following, which you may use without proof.
In orthogonal curvilinear coordinates (u_1, u_2, u_3), with corresponding unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and arc-length parameters h_1, h_2, h_3, the gradient of a scalar field f is given by

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\mathbf{e}_3.$$

The divergence of a vector field $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ is given by

$$\nabla.\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3F_1) \;+\; \frac{\partial}{\partial u_2}(h_3h_1F_2) \;+\; \frac{\partial}{\partial u_3}(h_1h_2F_3)\right] ,$$

and the curl of the same vector field is given by

$$\nabla\times\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} .$$

In spherical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, \phi)$, the arc-length parameters are $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$.
In cylindrical polar coordinates $(u_1, u_2, u_3) \equiv (\rho, \phi, z)$, the arc-length parameters are
$h_1 = 1$, $h_2 = \rho$, $h_3 = 1$.