Queen Mary
UNIVERSITY OF LONDON

B.Sc. EXAMINATION BY COURSE UNITS

Answers to 2009 MTH5102 Calculus III exam

1. (a) $\nabla V = -2x\mathbf{i} - 8y\mathbf{j} + 18z\mathbf{k}$ [M1A3]

(b) An (elliptic) hyperboloid (of two sheets) [accept just hyperboloid as the description for 2 marks: 1 each for elliptic and 2 sheets] [4]

(c) At $\mathbf{P}$, $\nabla V = -2\mathbf{i} - 8\mathbf{j} + 18\mathbf{k}$, so we get

$$-2x - 8y + 18z = 8$$

or equivalent, for the tangent plane. [4]

2.

$$\begin{aligned}
\mathbf{r} &= t^2\mathbf{i} + t^3\mathbf{j} + t\mathbf{k} \\
\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= 2t\mathbf{i} + 3t^2\mathbf{j} + \mathbf{k} \\
\mathbf{F} &= (2t^2 - t^4)\mathbf{i} + (t^3 + 2t)\mathbf{j} - t^5\mathbf{k} \ \text{ on the curve} \\
\mathbf{F}.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= 4t^3 - 2t^5 + 3t^5 + 6t^3 - t^5 = 10t^3. \\
\int_0^1 10t^3\mathrm{d}t &= \left[\frac{10t^4}{4}\right]_0^1 = \frac{5}{2}
\end{aligned}$$

Marks, per line above, 0,2,M2A2,M1A2,2. [11]

3.

$$\nabla\cdot\mathbf{F} = 1 + 3 + 2 = 6$$

so $\int_S \mathbf{F}.\mathrm{d}\mathbf{S} = \int \nabla\cdot\mathbf{F}\,\mathrm{d}V = 6.(4\pi a^3)/3 = 8\pi a^3$.
M1A3 for correct div, M3(can be implied) for Div Thm, A3 for result [10]

4. Both surfaces have a boundary consisting of the four edges of the square, taken in the same sense, i.e. clockwise with given surface normal.
2 for bounding curve, 3 for sense of it. [5]

The equality is due to Stokes's theorem, as both surfaces have the same bounding curve C, described in the same sense, and both integrals are equal to $\int_C \mathbf{F}.\mathrm{d}\mathbf{r}$. [3]

5. (a)

$$\begin{aligned}
\mathbf{F} &= z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k} \\
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & x^2 & y^2 \end{vmatrix} = 2y\mathbf{i} + +2z\mathbf{j} + 2x\mathbf{k} \neq 0
\end{aligned}$$

Hence no Φ is possible.
M2 for curl once only, A3 for curl. 1 for no Φ [6]

(b) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$
Easy to get $\nabla \times \mathbf{F} = \mathbf{0}$.
$\Phi = \frac{1}{3}(x^3 + y^3 + z^3) +$ constant.
[M2 for curl, may be above] A3 for curl, A3 for Φ [6]

6. In spherical polar coordinates, (using formulae given) [M1]

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi$$

and the divergence of $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi$ is [M2]

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2\sin\theta}\left[\frac{\partial(r^2\sin\theta F_r)}{\partial r} + \frac{\partial(r\sin\theta F_\theta)}{\partial\theta} + \frac{\partial(rF_\phi)}{\partial\phi}\right].$$

Hence putting these together we obtain [2]

$$\nabla^2\Phi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\Phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\right].$$

Inserting the given Φ we get

$$\begin{aligned}
\nabla^2\Phi &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta(2r\sin^2\theta\cos(2\phi))\right) + \frac{\partial}{\partial\theta}\left(\sin\theta(2r^2\sin\theta\cos\theta\cos(2\phi))\right)\right. \\
&\quad \left. + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}(-2r^2\sin^2\theta\sin(2\phi))\right)\right]
\end{aligned}$$

[This question continues overleaf . . .]

$$= \frac{1}{r^2\sin\theta}\left[6r^2\sin^3\theta\cos(2\phi) + 2r^2\cos(2\phi)(2\sin\theta\cos^2\theta - \sin^3\theta) - 4r^2\sin\theta\cos(2\phi)\right]$$
$$= \frac{r^2\cos(2\phi)}{r^2\sin\theta}(6\sin^3\theta + 4(1-\sin^2\theta)\sin\theta - 2\sin^3\theta - 4\sin\theta) = 0$$

They may directly put the given Φ into the grad and get

$$\nabla\Phi = 2r\sin^2\theta\cos(2\phi)\mathbf{e}_r + 2r\sin\theta\cos\theta\cos(2\phi)\mathbf{e}_\theta - 2r\sin\theta\sin(2\phi)\mathbf{e}_\phi.$$

A3 for the three bits of the grad (which may be implied in the ∇^2), A4 for the 4 bits of the second derivative, A1 for correct collection. [8]

7. This is not quite in the standard form: we need to divide by x^2 to get $f = 1$ and $g = (x-2)/x^2$. [1]
As $x \to 0$, g blows up so this is not an ordinary point. [2]
$xf = x$ and $x^2 g = x - 2$ have finite limits as $x \to 0$ so this is a regular singular point. [2]

Using given formulae, $f_0 = 0$, $f_1 = 1$, $g_0 = -2$, $g_1 = 1$, rest zero, so $c(c-1) - 2 = c^2 - c - 2 = (c-2)(c+1) = 0$ so $c = 2$ or $c = -1$. [2]

The recurrence relation gives $a_r[(r+c-2)(r+c+1)] + a_{r-1}[(r+c-1)+1] = a_r[(r+c-2)(r+c+1)] + a_{r-1}(r+c) = 0$.

Taking $c = -1$ we have $a_r[(r-3)r] + a_{r-1}(r-1) = 0$, [2]
so $a_1 = 0$ and the series terminates. The solution is a_0/x [2]
Note this is almost identical to an example given in lectures. Some students may do unnecessary work by substituting the Frobenius form into the equation. This is fine if they end with the right recurrence etc.

8. As this is an even function there are no sine terms. The cosine terms have coefficients

$$a_m = \frac{2}{\pi}\int_0^\pi e^x \cos mx\, dx.$$

Since we are given

$$\int_0^\pi e^x\cos(nx)\mathrm{d}x = \left[\frac{e^x\cos(nx)}{n^2}\right]_0^\pi - \frac{1}{n^2}\int_0^\pi e^x\cos(nx)\mathrm{d}x$$

we have

$$(n^2+1)\int_0^\pi e^x\cos(nx)\mathrm{d}x = [e^x\cos(nx)]_0^\pi = e^\pi\cos(n\pi) - 1$$

and using $\cos(n\pi) = (-1)^n$ we have

$$a_m = \frac{2}{\pi(m^2+1)}\left[e^\pi(-1)^m - 1\right].$$

[This question continues overleaf . . .]

(Note that there is no exceptional m to be dealt with separately.)
Substituting in the standard form of the Fourier series gives the result.
Marks: 2 for no sines. 3 for correct rearrangement of given result, 1 for $\cos(n\pi)$, 2 for a_n, 2 for final step [10]

9. (a) The solution must be of the form of a sum of the given terms.
Consider $x = 0$. Here we need

$$A_0(C_0 + D_0 y) + \sum A_n \left(C_n \cosh \frac{n\pi y}{a} + D_n \sinh \frac{n\pi y}{a} \right)$$
$$+ \sum a_n \left(c_n \cos \frac{n\pi y}{b} + d_n \sin \frac{n\pi y}{b} \right) = 0$$

This implies $A_i = a_i = 0$. Marks: M1 A2 [3]
Consider $y = 0$. Here we need

$$B_0 x C_0 + \sum \left(B_n \sin \frac{n\pi x}{a} \right) C_n + \sum \left(b_n \sinh \frac{n\pi x}{b} \right) c_n = 0$$

This implies $C_i = c_i = 0$. ($B_n = 0$ etc gives a zero solution.) [2]
Consider $y = b$. Here we need

$$B_0 x D_0 b + \sum B_n \sin \frac{n\pi x}{a} \left(D_n \sinh \left(\frac{n\pi b}{a} \right) \right) = 0$$

which gives $B_0 D_0 = 0 = B_n D_n$. [2]
The remaining terms are of the form $\sum b_n \sinh \frac{n\pi x}{b} \left(d_n \sin \frac{n\pi y}{b} \right)$ so the result comes from writing $K_n = b_n d_n$ [1]
Note this is essentially bookwork with x and y exchanged and the coefficients of x and y required given rather than found from boundary conditions. I will mark what are likely to be scrappy forms of the argument generously.

(b) On (0, b),

$$V_0 = \sum_{k=1}^{\infty} \frac{4V_0}{(2k+1)\pi} \sin(2k+1)\frac{\pi y}{b}.$$

Matching the above to the given Fourier series we have

$$\sum_{n=1}^{\infty} K_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b} = \sum_{k=1}^{\infty} \frac{4V_0}{(2k+1)\pi} \sin(2k+1)\frac{\pi y}{b}$$

Hence we need

$$K_n \sinh \frac{n\pi a}{b} = \frac{4V_0}{(2k+1)\pi}$$

for odd $n = 2k + 1$ only, and substituting back we get the solution

$$V = \sum_{k=1}^{\infty} \frac{4V_0 \sinh \frac{(2k+1)\pi x}{b}}{(2k+1)\pi \sinh \frac{(2k+1)\pi a}{b}} \sin \frac{(2k+1)\pi y}{b}$$

[5]

[Next question overleaf]

KEY OBJECTIVES of the course

The student should

1. Be able to do simple line and surface integrals. (E.g. Evaluate $\int \mathbf{F} \cdot d\mathbf{r}$ for a given vector field, with the path given in either parametric or non-parametric form.)
2. Be able to do simple manipulations involving gradient, divergence, and curl, and understand their geometrical/physical meaning.
3. Understand Stokes' theorem and the divergence theorem and be able to do simple problems applying these.
4. Be able to do simple manipulations in index notation, and switch between vector and index notation wherever necessary.
5. Understand three-dimensional cartesian, cylindrical, and spherical polar coordinates geometrically, and be able to express lines, surfaces, and volumes in coordinate or vector notation as appropriate.
6. Be able to obtain series solutions of differential equations using the Picard or Frobenius methods, including the Legendre, Bessel and Hermite functions.
7. Know the important properties of Fourier series and be able to compute coefficients.
8. Understand the variable-separation technique for PDEs and be able to solve simple problems with Laplace's equation in (at least) 2D Cartesian coordinates.

[End of examination paper]