Queen Mary
University of London

B.Sc. EXAMINATION BY COURSE UNITS

MAS204 Calculus III

Date and Time: 15th May 2008, 2.30 PM

The duration of this examination is 2 hours.

This paper has two sections and you should attempt both sections. Please read carefully the instructions given at the beginning of each section.

You are reminded of the following, which you may use without proof.
In orthogonal curvilinear coordinates (u_1, u_2, u_3), with corresponding unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and arc-length parameters h_1, h_2, h_3, the gradient of a scalar field f is given by

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\mathbf{e}_3.$$

The divergence of a vector field $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ is given by

$$\nabla.\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3F_1) \ + \ \frac{\partial}{\partial u_2}(h_3h_1F_2) \ + \ \frac{\partial}{\partial u_3}(h_1h_2F_3)\right] ,$$

and the curl of the same vector field is given by

$$\nabla\times\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} .$$

In spherical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, \phi)$, the arc-length parameters are $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$.
In cylindrical polar coordinates $(u_1, u_2, u_3) \equiv (\rho, \phi, z)$, the arc-length parameters are $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$.

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

SECTION A

You should attempt all questions. Marks awarded for correct answers are shown in square brackets next to the questions.

A1. (a) Find the gradient of $V = 9x^2 + y^2 - 4z^2$. [3]

(b) Sketch the surface $V = 6$ and describe its shape. [3]

(c) Find an equation for the line normal to this surface at $\mathbf{P} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. [4]

A2. Evaluate the integral $\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where $\mathbf{F} = 3y^2\mathbf{i} + 5z\mathbf{j} + 4x^2\mathbf{k}$ and $\mathcal{C}$ is the path going from $(0,0,0)$ to $(2,1,1)$ described in parametric form as $\mathcal{C} : \mathbf{r} = 2t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. [8]

A3. For each of the following vector fields $\mathbf{F}$, either find the most general Φ such that $\mathbf{F} = \nabla\Phi$, or show that no such Φ exists,

(a) $\mathbf{F} = x^2\mathbf{i} + y^3\mathbf{j} + z^2\mathbf{k}$, [4]

(b) $\mathbf{F} = xz\mathbf{i} + y^2\mathbf{j} + zx\mathbf{k}$. [4]

A4. Show that $\mathbf{F} = 2x\mathbf{i} + (y + x^2)\mathbf{j} + (3z + xy)\mathbf{k}$ has a constant divergence, and hence, using the Divergence Theorem, evaluate $\int_S \mathbf{F}.\mathrm{d}\mathbf{S}$ over the surface of the sphere of radius a centred at (1, 1, 1), where $\mathbf{S}$ is taken in the direction of the outward normal on the sphere. [8]

A5. Using index notation,

(a) simplify $\delta_{ij}\delta_{jk}\delta_{ki}$; [4]

(b) prove that $\nabla \times (\Phi\mathbf{F}) = \Phi\nabla \times \mathbf{F} - \mathbf{F} \times \nabla\Phi$, where Φ is a scalar field and $\mathbf{F}$ is a vector field. [5]

A6. Using Picard's method, find a power series solution of the equation [9]

$$u' = u + xu^2$$

with the initial conditions $u = 2$ and $x = 0$, up to terms of order x^3.

A7. Show that for $-\pi < x < \pi$, the odd function $f(x) = x$ obeys [8]

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(nx)}{n}.$$

[Next section overleaf]

SECTION B

Each question carries 20 marks. You may attempt all questions, but only marks for the best TWO questions will be counted.

B1. Let $\mathbf{a}$ be a constant vector and $\mathbf{r}$ denote the usual position vector. Using index notation, or otherwise,

(a) Show that [4]

$$\nabla(r^2(\mathbf{a}.\mathbf{r})) = 2(\mathbf{r}.\mathbf{a})\mathbf{r} + r^2\mathbf{a} \tag{1}$$

(b) Show that [11]
(i) $\nabla \times \mathbf{r} = \mathbf{0}$,
(ii) $\nabla r^2 = 2\mathbf{r}$,
(iii) $(\mathbf{b}.\nabla)\mathbf{r} = \mathbf{b}$ for any vector $\mathbf{b}$,
(iv) and that $(\mathbf{b}.\nabla)(r^2\mathbf{a}) = (2\mathbf{b}.\mathbf{r})\mathbf{a}$ for a constant vector $\mathbf{a}$.

[As usual, $(\mathbf{F}.\nabla)\mathbf{G}$, for a vector $\mathbf{G} = G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k}$, is defined to mean $(\mathbf{F}.\nabla G_1)\mathbf{i} + (\mathbf{F}.\nabla G_2)\mathbf{j} + (\mathbf{F}.\nabla G_3)\mathbf{k}$.]

You are given the identity

$$\nabla(\mathbf{F}.\mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F}.\nabla)\mathbf{G} + (\mathbf{G}.\nabla)\mathbf{F}\ . \tag{2}$$

Substituting $\mathbf{F} = r^2\mathbf{a}$ and $\mathbf{G} = \mathbf{r}$ into Equation (2), re-derive Equation (1). [5]

[In the final calculation, you may use the result stated in question A5, and you may assume that for any three vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}.\mathbf{C})\mathbf{B} - (\mathbf{A}.\mathbf{B})\mathbf{C}\ .]$$

B2. Find the line integral $\int_C \mathbf{F}.d\mathbf{r}$ for the vector field $\mathbf{F} = -y\mathbf{i} + 2x\mathbf{j} + y\mathbf{k}$ along the following curves

(a) C_1: $\mathbf{r} = (4 - t^2)\mathbf{i} + t\mathbf{j}, \qquad -2 \le t \le 2$

(b) C_2: the portion of the circle $y^2 + z^2 = 4$ in $x = 0$, $z \ge 0$, between $(0, 2, 0)$ and $(0, -2, 0)$. [You will need to first give a suitable parametrization for this curve.]

Identify C_1 and C_2 on a copy of the following diagram of the surface $x + y^2 + z^2 = 4$ in the region $x \ge 0$, $z \ge 0$.

[This question continues overleaf . . .]

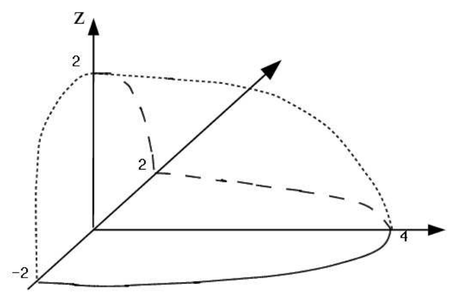

Figure 1: The diagram for question B.2.

Hence evaluate $\int_S \nabla \times F.\mathrm{d}\mathbf{S}$ over the surface $x + y^2 + z^2 = 4$ in the region $x \geq 0$, $z \geq 0$ with normal taken in the direction away from the origin.

B3. Show that $\int_S \nabla\Psi.\mathrm{d}\mathbf{S} = 0$ for any solution Ψ of Laplace's equation valid in a volume V with closed surface S.

Find the formula for $\nabla^2\Phi$ in spherical polar coordinates, and use this to show that $\Psi_1 = r^2 \sin\theta \cos\theta \cos\phi$, in spherical polar coordinates, is a solution of Laplace's equation.

By considering what property of $\cos\phi$ you used, explain without detailed calculation why the same must be true for $\Psi_2 = r^2 \sin\theta \cos\theta \sin\phi$.

By expressing Ψ_1 and Ψ_2 in Cartesians, explain why $\int_S \mathbf{F}.\mathrm{d}\mathbf{S}$ for a volume V with closed surface S is the same for $\mathbf{F} = \mathbf{F}_1 = (x + z)\mathbf{i} + (y + z)\mathbf{j} + (x + y)\mathbf{k}$ and for $\mathbf{F} = \mathbf{F}_2 = x\mathbf{i} + y\mathbf{j}$.

B4. Legendre's Equation is

$$(1 - x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \;-\; 2x\frac{\mathrm{d}y}{\mathrm{d}x} \;+\; \lambda y \;=\; 0.$$

Show that $x = 0$ is an ordinary point of this equation. Using the method of Frobenius, or otherwise, show that the equation has a polynomial solution if $\lambda = l(l+1)$, where l is a positive integer. Determine this solution for $l = 3$.

B5. Show that the Fourier series $S(x)$ of the function $f(x)$ which has the values

$$f(x) = \begin{cases} 0 & \text{for } -\pi \leq x \leq 0 \\ \sin x & \text{for } 0 < x \leq \pi \end{cases}$$

[This question continues overleaf . . .]

is

$$S(x) = \frac{2}{\pi}\left(\frac{1}{2} + \sum_{k=1}^{\infty} \frac{\cos 2kx}{1-4k^2}\right) + \frac{1}{2}\sin x.$$

Using $S(x)$ show that

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(4k^2-1)} = \frac{1}{2} - \frac{\pi}{4}.$$

B6. A function $F(x, y)$ satisfies Laplace's equation in the square $x, y \in [0, 1]$ with boundary values as indicated below.

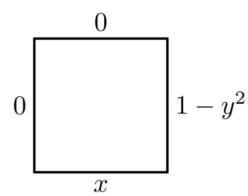

Figure 2: The diagram for question B.6

This $F(x, y)$ can be split into two functions

$$F(x, y) = f(x, y) + g(x, y)$$

such that at the corners $f(x, y) = 0$ and $g(x, y) = F(x, y)$.

(a) Find a suitable $g(x, y)$.

(b) Find $f(x, y)$ on the sides.

(c) Find $f(x, y)$ and hence $F(x, y)$.

[You may assume that the Fourier sine series for $y(1-y)$ for the range [0, 1] is

$$y(1-y) = \frac{8}{\pi^3} \sum_{p=0}^{\infty} \frac{\sin[(2p+1)\pi y]}{(2p+1)^3} \;.]$$

[End of examination paper]