B.Sc. EXAMINATION BY COURSE UNITS

Answers to 2008 MAS204 Calculus III exam

SECTION A

Answers in section A are cross-referenced to the Key Objectives (KO), in the order used in the course (Copy appended to these answers).

A1. {KO2 and KO5: similar problems in lectures and exercises}

(a) $\nabla V = 18x\mathbf{i} + 2y\mathbf{j} - 8z\mathbf{k}$ [3]

(b) An (ellipsoidal) hyperboloid (of one sheet) [accept just hyperboloid as the description] [3]

(c) At $\mathbf{P}$, $\nabla V = 18\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}$, so we get

$$\mathbf{r} = (1 + 18t)\mathbf{i} + (1 + 2t)\mathbf{j} + (1 - 8t)\mathbf{k}$$

or equivalent, for the normal line. [4]

Comment: Part a was well done but part b was not and the sketches often disagreed with the descriptions. In part c, many people did things like forget to substitute $\mathbf{P}$ *into* ∇V*, and a lot gave the tangent plane rather than the normal line.*

A2. {KO1} [method 3]

$$\begin{aligned}
\mathbf{r} &= 2t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \\
\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= 2\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \\
\mathbf{F} &= 3t^4\mathbf{i} + 5t^3\mathbf{j} + 16t^2\mathbf{k} \ \text{ on the curve} \\
\mathbf{F}.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= 6t^4 + 10t^4 + 48t^4 = 64t^4 \\
\int_0^1 64t^4\mathrm{d}t &= \left[\frac{64t^5}{5}\right]_0^1 = \frac{64}{5}
\end{aligned}$$

Marks, per line above, 0,1,1,2,1. [5]

Comment: Quite a lot of correct answers. Main errors were in the arithmetic, in using some other path, or in taking the upper limit of t *to be 2 rather than 1.*

A3. { KO 2 and 3}

[This question continues overleaf ...]

(a)

$$\begin{aligned}\mathbf{F} &= x^2\mathbf{i} + y^3\mathbf{j} + z^2\mathbf{k} \\ \Phi &= \frac{1}{3}x^3 + \frac{1}{4}y^4 + \frac{1}{3}z^3\end{aligned}$$

Various routes possible. [4]

(b)

$$\begin{aligned}\mathbf{F} &= xz\mathbf{i} + y^2\mathbf{j} + xz\mathbf{k} \\ \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & y^2 & xz \end{vmatrix} = (x-z)\mathbf{j} \neq 0\end{aligned}$$

Hence no Φ is possible. [4]

Comment: those who made a serious attempt did quite well.

A4. { KO 3}

$$\nabla \cdot \mathbf{F} = 2 + 1 + 3 = 6$$

so $\int_S \mathbf{F}.\mathrm{d}\mathbf{S} = \int \nabla \cdot \mathbf{F}\,\mathrm{d}V = 6.(4\pi a^3)/3 = 8\pi a^3$.
M2 for correct Div Thm, 3 for div, 3 for result [8]
Comment: a lot of people tried to directly evaluate $\int_S \mathbf{F}.\mathrm{d}\mathbf{S}$, without success. More than I would have hoped gave a vector rather than a scalar for $\nabla \cdot \mathbf{F}$.

A5. { KO 4}

(a) $\delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ik}\delta_{ki} = \delta_{kk} = 3$ [4]
Note: 2 for use of delta, 2 for $\delta_{kk} = 3$

(b) $[\nabla \times (\Phi\mathbf{F})]_i = \epsilon_{ijk}\partial_j(\Phi F_k) = \Phi\epsilon_{ijk}\partial_j F_k + \epsilon_{ijk}F_k\partial_j\Phi = [\Phi\nabla \times \mathbf{F} - \mathbf{F} \times \nabla\Phi]_i$ [5]

Comment: A lot of candidates were convinced that a question on index notation had to make use of the $\epsilon_{ijk}\epsilon_{ilm}$ identity so tried to bring it in in some way.

A6. { KO 6}

$$\begin{aligned}u_n &= 2 + \int_0^x (u_{n-1} + xu_{n-1}^2)\mathrm{d}x \\ u_1 &= 2 + \int_0^x (2+4x)\mathrm{d}x = 2 + 2x + \ldots \\ u_2 &= 2 + \int_0^x ((2+2x+\ldots) + x(2+2x\ldots)^2)\mathrm{d}x = 2 + 2x + 3x^2 \\ u_3 &= 2 + \int_0^x ((2+2x+3x^2\ldots) + x(2+2x\ldots)^2)\mathrm{d}x \\ &= 2 + \int_0^x ((2+2x+3x^2\ldots) + 4x + 8x^2\ldots)\mathrm{d}x = 2 + 2x + 3x^2 + \frac{11}{3}x^3\end{aligned}$$

Note: the exact solution is $1/(1 - x - \frac{1}{2}e^{-x})$
Marks: Method 3, u_1 1, u_2 2, u_3 3. [9]
Comment: candidates who got this wrong but knew the basic idea either stopped after too few iterations or were let down by their algebra or integration.

[Next question overleaf]

A7. { KO 7} As this is an odd function there are no cosine terms. The sine terms have

$$\begin{aligned} b_m &= \frac{2}{\pi}\int_0^\pi x\sin mx\,dx \\ &= \frac{2}{\pi}\left\{\left[\frac{-x\cos mx}{m}\right]_0^\pi + \int_0^\pi \frac{\cos mx}{m}\mathrm{d}x\right\} \\ &= \frac{2}{\pi}\left\{\frac{-\pi\cos m\pi}{m} + \left[\frac{\sin mx}{m^2}\right]_0^\pi\right\} \\ &= \frac{2(-1)^{m+1}}{m} \end{aligned}$$

Hence the result stated. [8]
Marks: 2 for cos, M2, A4 for sin bits
Comment: A fair number of good answers.

SECTION B

B1. All of these can be done by writing out components but it is messier.

(a) $\nabla(r^2(\mathbf{a}.\mathbf{r}))_i = \partial_i x_k x_k x_j a_j = 2\delta_{ik}x_k x_j a_j + x_k x_k \delta_{ij} a_j = 2x_i x_j a_j + x_k x_k a_i$ [4]

(b) $(\nabla\times\mathbf{r})_i = \epsilon_{ijk}\partial_j x_k = \epsilon_{ijk}\delta_{jk} = \epsilon_{ikk} = 0.$ [2]
$(\nabla r^2)_i = \partial_i(x_j x_j) = 2x_j\partial_i x_j = 2x_j\delta_{ij} = 2x_i$ [3]
$((\mathbf{b}.\nabla)\mathbf{r})_i = b_j\partial_j x_i = b_j\delta_{ji} = b_i.$ [2]
$[(\mathbf{b}.\nabla)(r^2\mathbf{a})]_i = b_j\partial_j(x_k x_k a_i) = 2a_i b_j x_k\partial_j x_k = 2a_i b_j x_k\delta_{jk} = 2a_i b_j x_j = [2(\mathbf{b}.\mathbf{r})\mathbf{a}]_i$ [4]
Note: parts (i) and (iii) done as bookwork: hence lower marks. }

Using the given identity, previous answers and Q A5

$$\begin{aligned} \nabla\left[(r^2\mathbf{a}).\mathbf{r}\right] &= r^2\mathbf{a}\times(\nabla\times\mathbf{r}) + \mathbf{r}\times(\nabla\times r^2\mathbf{a}) + (r^2\mathbf{a}.\nabla)\mathbf{r} + (\mathbf{r}.\nabla)r^2\mathbf{a} \\ &= r^2\mathbf{a}\times\mathbf{0} + \mathbf{r}\times(r^2\nabla\times\mathbf{a} - \mathbf{a}\times\nabla r^2) + r^2\mathbf{a} + 2(\mathbf{r}.\mathbf{r})\mathbf{a} \\ &= \mathbf{r}\times(-2\mathbf{a}\times\mathbf{r}) + r^2\mathbf{a} + 2r^2\mathbf{a} \\ &= -2[(\mathbf{r}.\mathbf{r})\mathbf{a} - (\mathbf{r}.\mathbf{a})\mathbf{r}] + 3r^2\mathbf{a} \\ &= 2(\mathbf{r}.\mathbf{a})\mathbf{r} + r^2\mathbf{a} \end{aligned}$$

[5]

Comment: See the comment on Q A4: the same problem occured here. Triply-occurring indices arose in some answers. Some got confused between vectors and scalars e.g. wrote things like $r^2 = (x^2, y^2, z^2)$. Handling of the differentiations was poor.

[This question continues overleaf . . .]

B2. (a) [6]

$$\begin{aligned} \mathbf{r} &= (4-t^2)\mathbf{i} + t\mathbf{j} \\ \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= -2t\mathbf{i} + \mathbf{j} \\ \mathbf{F} &= -t\mathbf{i} + 2(4-t^2)\mathbf{j} + t\mathbf{k} \quad \text{on the curve} \\ \mathbf{F}.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= 2t^2 + 2(4-t^2) = 8 \\ \int_{-2}^{2} 8\mathrm{d}t &= 32 \end{aligned}$$

(b) A suitable parametrization is $y = 2\cos\theta$, $z = 2\sin\theta$ [2]

$$\begin{aligned} \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta} &= -2\sin\theta\mathbf{j} + 2\cos\theta\mathbf{k} \\ \mathbf{F} &= -2\cos\theta\mathbf{i} + 0\mathbf{j} + 2\cos\theta\mathbf{k} \quad \text{on the curve} \\ \mathbf{F}.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= 0 + 0 + 4\cos^2\theta \\ \int_0^{\pi} 4\cos^2\theta\,\mathrm{d}\theta &= \tfrac{1}{2}4\pi = 2\pi \end{aligned}$$

[6]

C_1 is the curve in the $x-y$ plane in the figure, taken from -2.
C_2 is the curve in the $y-z$ plane in the figure, taken from 2.
The surface integral, by Stokes's theorem, is therefore the sum of the two previous results i.e. $2\pi + 32$ [6]

Comment: The line integrals were quite well done, especially the first of the two. Quite a few calculated $\nabla \times \mathbf{F}$, unnecessarily. Few did as requested for the sketch and quite a few misidentified the curves.

B3. {The calculation of ∇^2 in polars is bookwork.} $\int_S \nabla\Psi.\mathrm{d}\mathbf{S} = \int_V \nabla^2\Psi\,\mathrm{d}V = 0$ using the Divergence Theorem and the fact that Ψ solves Laplace. [4]

Now in spherical polar coordinates, (using formulae on the front sheet)

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi$$

and the divergence of $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi$ is

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2\sin\theta}\left[\frac{\partial(r^2\sin\theta F_r)}{\partial r} + \frac{\partial(r\sin\theta F_\theta)}{\partial\theta} + \frac{\partial(rF_\phi)}{\partial\phi}\right].$$

Hence putting these together we obtain [3]

$$\nabla^2\Phi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\Phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\right].$$

[This question continues overleaf . . .]

Inserting the given Ψ_1 we get

$$\begin{aligned}
\nabla^2\Psi_1 &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta(2r\sin\theta\cos\theta\cos\phi)\right)\right.\\
&\left. +\frac{\partial}{\partial\theta}\left(\sin\theta(r^2(\cos^2\theta-\sin^2\theta)\cos\phi)\right)+\left(\frac{1}{\sin\theta}(-r^2\sin\theta\cos\theta\cos\phi)\right)\right]\\
&= \frac{1}{r^2\sin\theta}\left[6r^2\sin^2\theta\cos\theta\cos\phi\right.\\
&\left. +r^2\cos\phi(\cos^3\theta-2\sin^2\theta\cos\theta-3\sin^2\theta\cos\theta)-r^2\cos\theta\cos\phi\right]\\
&= \frac{\cos\phi}{\sin\theta}(\sin^2\theta\cos\theta+\cos^3\theta-\cos\theta)=0
\end{aligned}$$

[5]

We only used $\dfrac{\partial^2\cos\phi}{\partial\phi^2}=-\cos\phi$ and $\sin\phi$ obeys the same equation. [2]

In Cartesians these are $\Psi_1 = xz$ and $\Psi_2 = yz$. [2]
The two vector fields $\mathbf{F}_1$ and $\mathbf{F}_2$ differ by $\mathbf{F}_1-\mathbf{F}_2 = z\mathbf{i}+z\mathbf{j}+(x+y)\mathbf{k} = \nabla(\Psi_1+\Psi_2)$ so the difference vanishes by the first result above. [4]
Comment: too few attempts at this to draw conclusions except that it was not popular.

B4. { All of this is bookwork. } This is not quite in the standard form: we would need to divide by $1-x^2$ to get $f=-2x/(1-x^2)$ and $g=\ell(\ell+1)/(1-x^2)$. As $x\to 0$ these tend to 0 and $\ell(\ell+1)$ respectively, so $x=0$ is an ordinary point. [2]

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty}(n+c)(n+c-1)a_nx^{(n+c-2)} - x^2\Big(\sum_{n=0}^{\infty}(n+c)(n+c-1)a_nx^{(n+c-2)}\Big)\\
&\quad -2x\Big(\sum_{n=0}^{\infty}(n+c)a_nx^{(n+c-1)}\Big)+\lambda\Big(\sum_{n=0}^{\infty}a_nx^{(n+c)}\Big)\\
&= \sum_{n=0}^{\infty}(n+c)(n+c-1)a_nx^{(n+c-2)} - \sum_{n=0}^{\infty}(n+c)(n+c-1)a_nx^{(n+c)}\\
&\quad -2\sum_{n=0}^{\infty}(n+c)a_nx^{(n+c)}+\sum_{n=0}^{\infty}\lambda a_nx^{(n+c)}\\
&= \sum_{n=0}^{\infty}(n+c)(n+c-1)a_nx^{(n+c-2)} + \sum_{n=0}^{\infty}[\lambda-(n+c)(n+c+1)]a_nx^{(n+c)}\\
&= \sum_{n=-2}^{\infty}(n+c+2)(n+c+1)a_{n+2}x^{(n+c)} + \sum_{n=0}^{\infty}[\lambda-(n+c)(n+c+1)]a_nx^{(n+c)}\\
0 &= \sum_{n=-2}^{\infty}\{(n+c+2)(n+c+1)a_{n+2}+[\lambda-(n+c)(n+c+1)]a_n\}x^{(n+c)}=0.
\end{aligned}$$

[This question continues overleaf . . .]

[M3,A3]

Taking the $n=-2$ term in the sum we find [2]

$$(c)(c-1)a_0x^{c-2}=0$$

and since $a_0 \neq 0$ this implies $c=0$ or $c=1$. [In fact the values of c at any ordinary point of any equation are always $c=0$ and $c=1$.]

Taking the coefficient of $x^{(r+c)}$ for $r>-2$ we have [2]

$$(r+c+2)(r+c+1)a_{r+2}+[\lambda-(r+c)(r+c+1)]a_r=0\ .$$

In the case $c=1$, we have

$$(r+3)(r+2)a_{r+2}=[(r+2)(r+1)-\lambda]a_r$$

which in particular (for $r=-1$) gives $a_1=0$. All higher a_n with odd n will then also be zero. The series will terminate if ℓ is an odd integer: taking $r=\ell-1$ gives [3]

$$(\ell+2)(\ell+1)a_{\ell+1}=[(\ell+1)\ell-\lambda]a_{\ell-1}\ ,$$

so we see that if $\lambda=(\ell+1)\ell$, $a_{\ell+1}=0$. In this case the $c=1$ series becomes just a polynomial in odd powers of x, with highest power x^ℓ. For example for $\ell=3$, $6a_2=-10a_0$, $20a_4=0a_2$, and [3]

$$u=a_0x(1-\tfrac{5}{3}x^2).$$

If $c=0$, we have

$$(r+2)(r+1)a_{r+2}=[(r+1)r-\lambda]a_r$$

and in particular a_1 ($r=-1$) can have any value, meaning we can add a multiple of the series with $c=1$. Taking just the even terms, we see that if $\lambda=(\ell+1)\ell$ where ℓ is a positive even integer, $a_{\ell+2}=0$. So we again have a polynomial, this time of even powers of x, with highest power x^ℓ. [2]

Thus for the Legendre equation with $\lambda=(\ell+1)\ell$ where ℓ is an integer, we will always get one solution which is a polynomial of degree ℓ.

Comment: Not as well done as I would have hoped. This suggests the presentation in lectures needs to be simplified.

B5. {Unseen but using basic Fourier properties. Similar to last year's. }

$f-\frac{1}{2}\sin x$ is even so only has a cosine series. [5]

The coefficients in the Fourier cosine series of $\frac{1}{2}\sin x$ are given by

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi \tfrac{1}{2}\sin x\cos nx\mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^\pi(\sin(n+1)x+\sin(1-n)x)\mathrm{d}x \end{aligned}$$

[This question continues overleaf . . .]

$$
\begin{aligned}
&= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(1-n)x}{1-n}\right]_0^\pi \\
&= \frac{1}{2\pi}\left[\frac{1-(-1)^{n+1}}{n+1} + \frac{1-(-1)^{1-n}}{1-n}\right] \\
&= \frac{1}{2\pi}\frac{(1-(-1)^{1+n})(1-n+1+n)}{1-n^2}
\end{aligned}
$$

if $n \neq 1$, [M2,A2+3]
and $\frac{2}{\pi}\int_0^\pi \sin x \cos x \mathrm{d}x = \frac{1}{2\pi}[-\cos 2x]_0^\pi = 0$ if $n = 1$. [1]
Thus $a_n = 0$ for all odd n. [1]
For even $n = 2k \geq 0$ we get $2/(1-4k^2)\pi$ which easily gives the required $S(x)$. [1]

{If candidates did not see this way of doing it, they calculated

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^\pi \sin x \sin nx \mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^\pi (\cos(n-1)x - \cos(n+1)x)\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^\pi \\
&= 0
\end{aligned}
$$

if $n \neq 1$. If $n = 1$ we have

$$
b_1 = \frac{1}{\pi}\int_0^\pi \sin^2 x \mathrm{d}x = \frac{1}{\pi}\frac{\pi}{2} = \tfrac{1}{2}\}
$$

Evaluation at $x = \pi/2$ gives [5]

$$
\begin{aligned}
1 &= \frac{2}{\pi}\left(\frac{1}{2} + \sum_{k=1}^\infty \frac{\cos k\pi}{1-4k^2}\right) + \frac{1}{2}. \\
&= \frac{2}{\pi}\left(\frac{1}{2} + \sum_{k=1}^\infty \frac{(-1)^k}{1-4k^2}\right) + \frac{1}{2}. \\
\tfrac{1}{2} &= \frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^\infty \frac{(-1)^k}{1-4k^2}. \\
\frac{(-1)^k}{4k^2-1} &= \frac{1}{2} - \frac{\pi}{4}
\end{aligned}
$$

Comment: Rather well done, comparatively

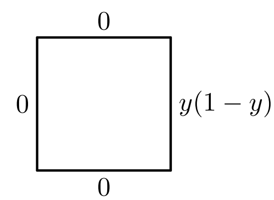
B6. We use the standard trick for eliminating corners: set

$$
g(x,y) = \alpha + \beta x + \gamma y + \delta xy
$$

[This question continues overleaf . . .]

and solve for the corner values. $g(0,\,0) = 0 \Rightarrow \alpha = 0$
$g(1,\,0) = 1 \Rightarrow \beta = 1$
$g(0,\,1) = 0 \Rightarrow \gamma = 0$
$g(1,\,1) = 0 \Rightarrow \delta = -1$
which gives $g(x, y) = x - xy$. [M3,A4]
(Note that $g = 0$ on $x = 0$ and $y = 1$, $g = x$ on $y = 0$ and $g = 1 - y$ on $x = 1$.)

The boundary values of $f(x, y)$ are simply the boundary values of $F(x, y) - g(x, y)$, or as shown below [3]



A solution of the form

$$f(x, y) = \sum_{n=1}^{\infty} b_n \sinh(n\pi x/a) \sin(n\pi y/a)$$

will work. (It satisfies Laplace's equation, and is zero for $x = 0$, $x = a$ and $y = 0$ as required.) and here $a = 1$. [3]

We are given that

$$y(1 - y) = \frac{8}{\pi^3} \sum_{p=0}^{\infty} \frac{\sin[(2p + 1)\pi y]}{(2p + 1)^3}.$$

Thus b_n is such that [3]

$$\sum_{n=1}^{\infty} b_n \sin(n\pi y) \sinh(n\pi) = \frac{8}{\pi^3} \sum_{p=0}^{\infty} \frac{\sin[(2p + 1)\pi y]}{(2p + 1)^3}.$$

Hence the solution is [2]

$$f(x, y) = \sum_{p=1}^{\infty} \frac{8}{(2p + 1)^3 \pi^3} \frac{\sinh((2p + 1)\pi x)}{\sinh((2p + 1)\pi)} \sin((2p + 1)\pi y)\ .$$

and the overall solution is $f + g$. [2]
Comment: relatively few serious attempts

[Next section overleaf]

KEY OBJECTIVES of the course

The student should

1. Be able to do simple line and surface integrals. (E.g. Evaluate $\int \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ for a given vector field, with the path given in either parametric or non-parametric form.)
2. Be able to do simple manipulations involving gradient, divergence, and curl, and understand their geometrical/physical meaning.
3. Understand Stokes' theorem and the divergence theorem and be able to do simple problems applying these.
4. Be able to do simple manipulations in index notation, and switch between vector and index notation wherever necessary.
5. Understand three-dimensional cartesian, cylindrical, and spherical polar coordinates geometrically, and be able to express lines, surfaces, and volumes in coordinate or vector notation as appropriate.
6. Be able to obtain series solutions of differential equations using the Picard or Frobenius methods, including the Legendre, Bessel and Hermite functions.
7. Know the important properties of Fourier series and be able to compute coefficients.
8. Understand the variable-separation technique for PDEs and be able to solve simple problems with Laplace's equation in (at least) 2D Cartesian coordinates.

[End of examination paper]