Queen Mary
University of London

B.Sc. EXAMINATION BY COURSE UNITS

MAS204 Calculus III: first sit paper

Date and Time: 15 May 2007, 2.30 PM

The duration of this examination is 2 hours.

This paper has two sections and you should attempt both sections. Please read carefully the instructions given at the beginning of each section.

This question paper must not be removed from the examination room.

You are reminded of the following, which you may use without proof.
In orthogonal curvilinear coordinates $(u_1,\, u_2,\, u_3)$, with corresponding unit vectors $\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3$ and arc-length parameters $h_1,\, h_2,\, h_3$, the gradient of a scalar field f is given by

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\mathbf{e}_3.$$

The divergence of a vector field $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ is given by

$$\nabla.\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3F_1) \;+\; \frac{\partial}{\partial u_2}(h_3h_1F_2) \;+\; \frac{\partial}{\partial u_3}(h_1h_2F_3)\right] ,$$

and the curl of the same vector field is given by

$$\nabla\times\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} .$$

In spherical polar coordinates $(u_1,\, u_2,\, u_3) \equiv (r,\, \theta,\, \phi)$, the arc-length parameters are $h_1 = 1,\, h_2 = r,\, h_3 = r\sin\theta$.
In cylindrical polar coordinates $(u_1,\, u_2,\, u_3) \equiv (\rho,\, \phi,\, z)$, the arc-length parameters are $h_1 = 1,\, h_2 = \rho,\, h_3 = 1$.

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

SECTION A

You should attempt all questions. Marks awarded for correct answers are shown next to the questions.

A1. (a) Find the gradient of $V = x^2 + y^2 - z$.

(b) Sketch the surface $V = 1$.

(c) Find an equation for the tangent plane to this surface at $\mathbf{P} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. [10]

A2. Find the line integral $\int_C \mathbf{F}.\mathrm{d}\mathbf{r}$ for the vector field $\mathbf{F} = (x - y)\mathbf{i} + 2xy\mathbf{j} + y\mathbf{k}$ along the following curves

(a) C_1: $\mathbf{r} = (4 - t^2)\mathbf{i} + t\mathbf{j}, \qquad -2 \leq t \leq 2$

(b) C_2: the portion of the circle $y^2 + z^2 = 4$ in $x = 0$, $z > 0$, between $(0, 2, 0)$ and $(0, -2, 0)$. [8]

A3. Explain why the integral $\int_S \nabla \times \mathbf{F}.\mathrm{d}\mathbf{S}$ over the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, is the same as $\int_S \nabla \times \mathbf{F}.\mathrm{d}\mathbf{S}$ over the disc $x^2 + y^2 \leq a^2$, $z = 0$, assuming that the normal to both surfaces is taken in the positive z direction and that $\nabla \times \mathbf{F}$ is piecewise continuous.

Evaluate $\oint_C \mathbf{F}.\mathrm{d}\mathbf{r}$ round the circle C given by $x^2 + y^2 = a^2$, $z = 0$, taken in the counter-clockwise sense in the x, y plane as seen from positive z, where $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$, by first converting this to an appropriate surface integral. [9]

A4. Prove that

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G}.\nabla)\mathbf{F} - \mathbf{G}(\nabla \cdot \mathbf{F}) - (\mathbf{F}.\nabla)\mathbf{G},$$

where by definition $(\mathbf{a}.\nabla)\mathbf{b}$ means the vector whose ith component is $a_j \partial b_i / \partial x_j$. [7]

[You may quote without proof the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.]

A5. Show that the Fourier series of the odd function $f(x)$ which has the values

$$f(x) = \begin{cases} -1 & \text{for } -\pi \leq x \leq 0 \\ 1 & \text{for } 0 < x \leq \pi. \end{cases}$$

is

$$S(x) = \sum_{p=0}^{\infty} \frac{4\sin((2p+1)x)}{(2p+1)\pi}.$$

Using Parseval's theorem, show that

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

[9]

[This question continues overleaf . . .]

[You are reminded that, with the usual notation for Fourier series, Parseval's theorem states that under appropriate conditions

$$\int_{-\pi}^{\pi} f^2\,\mathrm{d}x = \tfrac{1}{2}\pi a_0^2 + \pi \sum_{n=1}^{\infty}(a_n^2 + b_n^2).]$$

A6. What is the condition on $L(x,\,y,\,y')$, where $y' = \mathrm{d}y/\mathrm{d}x$, that ensures that the paths $y(x)$ which extremize the integral $\int L(x,\,y,\,y')\mathrm{d}x$ obey the equation $\partial L/\partial y' =$ constant?

If $L = x^{2n}y'^2$, where n is a constant, find the extremizing paths. Under what conditions can they join points at positive and negative values of x? [8]

A7. Given that $\Phi = g(x)\sinh(2\pi y)$ is a solution of $\nabla^2\Phi = 0$ (Laplace's equation), find the most general form of $g(x)$.

Starting with your general form, find the particular $g(x)$ such that Φ obeys the following conditions on the rectangle $0 \le x \le 1$, $0 \le y \le 2$:
$\Phi = 0$ on $x = 0$, on $y = 0$ and on $x = 1$,
$\Phi = \sin(2\pi x)$ on $y = 2$. [9]

SECTION B

Each question carries 20 marks. You may attempt all questions, but only marks for the best TWO questions will be counted.

B1. A vector field $\mathbf{F}$ satisfies

$$\nabla \times \mathbf{F} = f\mathbf{F}.$$

If f is constant, $\mathbf{F}$ is called a Beltrami field.

Show that

(a) $(\nabla \times \mathbf{F}) \times \mathbf{F} = \mathbf{0}$,

(b) if $\nabla \cdot \mathbf{F} = 0$ then $\mathbf{F}.\nabla f = 0$, and

(c) if $\mathbf{F}$ is a Beltrami field, so is $\nabla \times \mathbf{F}$.

Find the appropriate value of f such that the field

$$\mathbf{F} = A\sin(y^3)\mathbf{i} + A\cos(y^3)\mathbf{k},$$

where A is a constant, obeys $\nabla \times \mathbf{F} = f\mathbf{F}$. Show further that $\nabla \cdot \mathbf{F} = 0$.

[Next question overleaf]

B2. In spherical polar coordinates the vector field $\mathbf{F}$ is given by

$$\mathbf{F} = r(\sin\theta + \cos\theta)\mathbf{e}_r + 2r\sin\theta\mathbf{e}_\theta.$$

Verify the Divergence Theorem for this vector field over a sphere of radius a centered at the origin.

Using the Divergence Theorem or otherwise, evaluate $\int \mathbf{F}.\mathrm{d}\mathbf{S}$ over the surface of the hemisphere $x^2 + y^2 + z^2 \leq a^2$, $z \geq 0$.

B3. A vector field $\mathbf{F}$ has the form $G(x, y)\mathbf{i} + H(x, y)\mathbf{j}$ where

$$G = \sum_{n=1}^{\infty} A_n \cos(n\pi x/a)\sinh(n\pi y/a), \qquad H = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a)\cosh(n\pi y/a),$$

and the A_n are constants.

(a) Show that this field is conservative and give the Φ such that $\mathbf{F} = \nabla\Phi$.

(b) Show that this Φ obeys Laplace's equation.

(c) Find the values of the A_n for which $\Phi = 0$ on $x = 0$, $x = a$, and $y = 0$, and for which $\partial\Phi/\partial y$ (i.e. H) is $\frac{1}{2}a - |\frac{1}{2}a - x|$ on $y = b$.

[You may assume the Fourier sine series

$$\tfrac{1}{2}\pi - |\tfrac{1}{2}\pi - x| = \frac{4}{\pi}\sum_{p=1}^{\infty} \frac{(-1)^p \sin((2p+1)x)}{(2p+1)^2}.]$$

B4. Show that the Fourier series $S(x)$ of the function $f(x)$ which has the values

$$f(x) = \begin{cases} 0 & \text{for } -\pi \leq x \leq 0 \\ \cos x & \text{for } 0 < x \leq \pi \end{cases}$$

is

$$S(x) = \tfrac{1}{2}\cos x + \frac{4}{\pi}\sum_{p=1}^{\infty} \frac{p\sin(2px)}{(4p^2 - 1)}.$$

What is the value of $S(x)$ at $x = \pi$?

Using $S(x)$ show that

$$\sum_{s=0}^{\infty} \frac{(2s+1)(-1)^s}{(4(2s+1)^2 - 1)} = \frac{1}{3} - \frac{3}{35} + \frac{5}{99} \ldots = \frac{\pi\sqrt{2}}{16}.$$

[Next question overleaf]

B5. $y(x)$ extremizes the integral

$$I[y] = \int_a^b L(x,\, y,\, y')\, \mathrm{d}x \ ,$$

with $y(a)$ and $y(b)$ fixed and L given. Write down the equation which y must satisfy.

Show that if $L = L(y,\, y')$ (i.e. L has no explicit x dependence) then there is a first integral

$$y' \frac{\partial L}{\partial y'} - L = \text{constant.}$$

If $L = [(y')^2 - k^2 y^2]$ where k is a constant, find the curve $y(x)$ extremizing $I[y]$ and passing through the points $(0, 0)$ and $(\pi/2k, 5)$.

B6. In cylindrical polar coordinates the z-independent separable solutions of Laplace's equation $\nabla^2 \Phi = 0$, written as $R(\rho)S(\phi)$, can be shown to obey

$$\frac{\rho}{R} \frac{\mathrm{d}}{\mathrm{d}\rho} \left(\rho \frac{\mathrm{d}R}{\mathrm{d}\rho} \right) = -\frac{1}{S} \frac{\mathrm{d}^2 S}{\mathrm{d}\phi^2} = \lambda,$$

where λ is the separation constant.

(a) Show that if Φ is single-valued on circles centred on the z-axis, only values $\lambda = m^2 \geq 0$ are allowed, and

(b) give the most general corresponding $S(\phi)$ and $R(\rho)$.

Find the solution of Laplace's equation which is bounded at $\rho = 0$ and takes the values $4 \cos^2 \phi + \sin(3\phi)$ at $\rho = 2$.

[End of examination paper]