Calculus III
Answers to and comments on the 2007 test
Prof. M.A.H. MacCallum

Referring to the test version as on the course web page, the answers were

1	2
2	3
3	1
4	2
5	3
6	1
7	4
8	3
9	2
10	1

Comments and details

116 scripts were received and the average score was approximately 54%. This was a bit disappointing, although in line with other courses, as I had hoped I had made the test somewhat easier this year. I expected the scores on index notation not to be so good: it was recently taught and new to the test. What I did not expect was such a low score on the question about equations of surfaces. However, 5 questions were got right by over 75 candidates: in particular the questions on grad, div and curl were generally well done which is encouraging. Also, the number of candidates below 40% was not very high and some of those have been doing well in coursework so were presumably just unlucky in the test.

Here are details for individual questions. In the performance analysis answer 0 means no answer, 1 means the first choice, and so on.

1. Answers 1 and 3 are equations for a line through **a** in the direction **n**.

Performance: 0:11, 1:11, 2:77, 3:17. Generally well answered.

2. The similar equations for a sphere or paraboloid respectively would be $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = z + 4$.

Performance: 0:8, 1:53, 2:19, 3:36. I was really surprised so many thought this was a sphere – even more than made a similar error in last year's exam question A1.

3. The pictures are just the ones in lectures with all the arrows reversed.

Performance: 0:7, 1:81, 2:18, 3:10. Well answered.

4. The answer must be a scalar, so choice 1 cannot be correct. The wrong answer 3 arises from taking

$$\frac{\partial F_x}{\partial x} + \frac{\partial F_x}{\partial y} + \frac{\partial F_x}{\partial z} + \frac{\partial F_y}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_y}{\partial z} + \frac{\partial F_z}{\partial x} + \frac{\partial F_z}{\partial y} + \frac{\partial F_z}{\partial z}$$

rather than the correct

$$\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Performance: 0:4, 1:23, 2:81, 3:8. Well answered.

5. In answer 1, index m appears three times. In answer 2, the free index on the left is m and on the right k.

Performance: 0:47, 1:5, 2:20, 3:44. Pleasing that the majority of those who did not opt out got this right.

6. The gradient must be a vector, so answers 3 and 4 are wrong (they arise from taking answers 1 and 2 and adding the components together). $\nabla V = (y^2 + 3z)\mathbf{i} + 2xy\mathbf{j} + 3x\mathbf{k}$. Answer 2 arises from an incorrect evaluation of this at the point $(2, -1, 1)$.

Performance: 0:2, 1:88, 2:3, 3:19, 4:4. The best done of all the questions. Main error was in giving a scalar rather than a vector.

7. Answer 1 arises from taking

$$\frac{\partial F_x}{\partial x}\mathbf{i} + \frac{\partial F_y}{\partial y}\mathbf{j} + \frac{\partial F_z}{\partial z}\mathbf{k} \ .$$

Answer 3 is some version of confusing divergence and curl (could have been made a simpler form of an error). Answers 2 and 4 both come from a correct start with

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y - z^2 & yz \end{vmatrix} .$$

In the evaluation, only the $-z^2$ and yz terms, differentiated respectively with respect to z and y, can contribute. The wrong answer 2 comes from losing track of a minus.

Performance: 0:4, 1:4, 2:27, 3:3, 4:78. Well answered. Obviously most of those who got it wrong lost a sign in the evaluation.

8. On the curve $\mathrm{d}\mathbf{r} = \mathbf{r}_t\,\mathrm{d}t = (-\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j})\mathrm{d}t$ and $\mathbf{F} = (2\sin t)\mathbf{i} + 2(\cos t)\mathbf{k}$, so the dot product is $-2\sin^2 t\,\mathrm{d}t$. The limits on t are 0 and π. The integral becomes

$$\int_0^\pi (-2\sin^2 t)\mathrm{d}t = \int_0^\pi (\cos 2t - 1)\mathrm{d}t = [\tfrac{1}{2}\sin 2t - t]_0^\pi = -\pi \ .$$

The wrong answers arise in various ways e.g. failing to differentiate $\mathbf{r}$, wrong limits on t, incorrect differentiation or integration. This was the question involving the most calculation. (It is in fact easy to do by not using the parametrization but noting that the answer is $\int_1^{-1} y\,\mathrm{d}x$ for the curve which is minus the area under the curve, i.e. minus half the area of the ellipse which itself is 2π, so again we get $-\pi$.)

Performance: 0:25, 1:12, 2:43, 3:19, 4:17. Disappointing seeing I had deliberately set an example very close to last year's line integral (virtually the same with the coordinates permuted). I'm not clear which mistakes candidates were making – one of those examples where a traditional exam would have been more revealing than a multiple choice format.

9. Answer 2 incorrectly states Stokes's theorem. It is wrong because the sense of the curve is clockwise rather than anticlockwise relative to $\mathbf{k}$: you can see this because $t = 0$ is at $a\mathbf{j}$ and $t = \frac{1}{2}\pi$ is at $a\mathbf{i}$. Answer 1 states the Divergence Theorem, and answer 4 states a standard result for a conservative field. Answer 3 follows from Stokes's theorem since $\mathcal{H}$ and $\mathcal{D}$ are surfaces with compatible normals and the same bounding curve so both area integrals are equal to the same line integral: the formula arises since on $\mathcal{D}$, $\mathrm{d}\mathbf{S} = \mathbf{k}\,\mathrm{d}x\,\mathrm{d}y$.

Performance: 0:48, 1:7, 2:21, 3:32, 4:8. I had hoped for more correct answers having emphasized in my advice on revision the need to understand the directions in the various theorems.

10. Since answers 1 and 3 contradict each other, one of them must be the wrong one. (This intentionally made the question easier, as this is a topic new to the mid-term test. The hard bit is in correctly changing the names of the indices from those in the given expression.).

$$\epsilon_{mki}\epsilon_{ilm} = \epsilon_{mki}\epsilon_{mil} = \delta_{ki}\delta_{il} - \delta_{ii}\delta_{kl} = \delta_{kl} - 3\delta_{kl} \ ,$$

so answer 3 is a correct formula and therefore the wrong answer. To see that answer 2 is equal to the given expression, just use the index substitution property of delta and the cyclic symmetry property of epsilon to get $\delta_{ij}\epsilon_{mkj}\epsilon_{mil} = \epsilon_{mki}\epsilon_{mil} = \epsilon_{mki}\epsilon_{ilm}$. To check answer 4, note $-\epsilon_{mik}\epsilon_{mil} = \epsilon_{mki}\epsilon_{mil}$.

Performance: 0:57, 1:21, 2:17, 3:11, 4:10. This was the one where I most expected a poor result. Next year they'll all get it right! (I hope.)