S357/D

The Open University

Course Examination 2004

Space, Time and Cosmology

Tuesday 12 October 2004 2.30pm – 5.30pm

Time allowed: 3 hours

This paper is divided into **TWO** parts, I and II. Part I carries 48 per cent of the total marks, and Part II carries 52 per cent. You are advised to spend roughly equal time on each part. Remember to allow time for reading the question paper first, and your answers afterwards. You will not be given extra time to do this.

You should answer **ALL** the questions in Part I, and **FOUR** questions from Part II.

Record your answers to Part I in the spaces provided in the green question and answer booklet inserted in this question paper. Answer **each** question attempted from Part II in a **separate** answer book (i.e. use **four** answer books in all). *It is most important that you indicate in the box provided on the front of each answer book the number of the question which you have answered in that book.*

At the end of the examination

1. Make sure that you have written your personal identifier and examination number on Part I of the question paper and on **each** answer book used. **Failure to do so will mean that your work cannot be identified.**
2. Complete the grid on the front of the enclosed green booklet containing Part I.
3. Put all your used answer books and your question paper together with your signed desk record on top. Fix them all together with the paper fastener provided.

Note: Part I of this paper is provided as a separate insert.

Copyright © 2004 The Open University

S357/D

PART I **AT THE END OF THE EXAMINATION, ATTACH THIS PART TO THE FRONT OF THE ANSWER BOOKS FOR PART II USING THE PAPER FASTENER PROVIDED.**

Examination No.								
Personal Identifier								

Please indicate on this grid which questions from Part II you have answered. (Details of Part I are *not* required.)

Part II			

Instructions for Part I

(*i*) Part I carries 48 per cent of the total marks, and you are advised to spend about **90 minutes** on it.

(*ii*) For each of the twelve questions, fill in the answer box(es) or space provided as indicated in the question.

(*iii*) Attempt as many questions as you can. There are **NO** penalties in Part I.

Copyright © 2004 The Open University

Question 1 (*4 marks*) Two particles, A and B experience the same constant force **F** and are found to have the following position vectors (in SI units) for times $t \geqslant 0$ in an inertial frame S:

$$\mathbf{x}_{\mathrm{A}}(t) = (t + 2,\ 3,\ t^2 + 2t - 2)$$
$$\mathbf{x}_{\mathrm{B}}(t) = (3,\ 4 - t,\ 2t^2 - 1).$$

Use Newtonian mechanics to answer parts (a) to (d), assuming that SI units are used throughout. For each of (a) to (d) select **ONE** item from the key for Q1.

(a) What is the numerical value of the speed of particle A at time $t = 0$? ☐

(b) What is the numerical value of the time when the particles collide? ☐

(c) What is the numerical value of the particles' relative speed when they collide? ☐

(d) What is the numerical value obtained by dividing the mass of particle A by the mass of particle B? ☐

KEY for Q1

A 1/4

B 1/2

C 1

D $\sqrt{2}$

E 4/3

F 3/2

G $\sqrt{5}$

H 2

Question 2 (*4 marks*) A particle of mass m is ejected from the origin of a Cartesian coordinate system at time $t = 0$. At some later time, $t > 0$, the particle hits a wall. At the time of the impact, the position, velocity and acceleration of the particle are given by $\mathbf{x} = (x^{\mathbf{1}}, x^{\mathbf{2}}, x^{\mathbf{3}})$, $\mathbf{v} = (v^{\mathbf{1}}, v^{\mathbf{2}}, v^{\mathbf{3}})$ and $\mathbf{a} = (a^{\mathbf{1}}, a^{\mathbf{2}}, a^{\mathbf{3}})$, respectively.
Note that the components of these vectors are identified by **bold** superscripts and that ordinary, unbold superscripts are powers, as usual.

(*a*) Which **TWO** of the items in the key for Q2 are invariant under a change in the origin of the Cartesian coordinate system? ☐☐

(*b*) Which **TWO** of the items in the key for Q2 are invariant under a recalibration of the time-scale from seconds to years? ☐☐

(*c*) Which **TWO** of the items in the key for Q2 are invariant under *all* rotations of the Cartesian coordinate system? ☐☐

(*d*) Which **TWO** of the items in the key for Q2 are **NOT** invariant under a shift in the origin of time such that the particle is ejected at $t = T > 0$? ☐☐

KEY for Q2

A $m(x^{\mathbf{1}} - a^{\mathbf{1}}t^2)$

B $(\mathbf{a} \times \mathbf{x}) \cdot (\mathbf{v} \times \mathbf{x})/|\mathbf{v}|$

C $m(a^{\mathbf{3}}v^{\mathbf{1}} - a^{\mathbf{1}}v^{\mathbf{3}})$

D $(\mathbf{a} \cdot \mathbf{x})/(\mathbf{v} \cdot \mathbf{v})$

E $m(v^{\mathbf{1}} - a^{\mathbf{1}}t)$

Question 3 (*4 marks*) Which **FOUR** of the statements in the key for Q3 are true? ☐☐☐☐

KEY for Q3

A Kepler's second law (equal areas) implies that the gravitational force obeys an inverse square law.

B Kepler's second law (equal areas) implies that the gravitational force is central.

C Kepler's third law implies that for planets in circular orbits, the orbital speed decreases as the orbital radius increases.

D Kepler's third law implies that for planets in circular orbits, the orbital speed increases as the orbital radius increases.

E Newton's law of universal gravitation implies that the gravitational force a particle A exerts on another particle B differs from the gravitational force that B exerts on A, even though the two forces have the same magnitude.

F Newton's law of universal gravitation implies that there would be no change in the force exerted on an orbiting satellite by a homogeneous spherical planet of mass m if the planet suddenly shrank to a much smaller size, provided it remained homogeneous and its mass was unchanged.

G Newton's first law of motion holds true in all frames of reference, regardless of whether or not they are inertial frames.

H Newton's second law of motion holds true in all frames of reference, regardless of whether or not they are inertial frames.

Question 4 (*4 marks*) Which **FOUR** of the statements in the key for Q4 are true? ☐☐☐☐

KEY for Q4

A The Lorentz force law is invariant under Galilean transformation.

B The Lorentz force law can provide an operational definition of **E** and **B**.

C If a charged particle moves with constant velocity **v** through a region of uniform electric and magnetic fields, and if the only force on the particle is the Lorentz force, then $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ throughout that region.

D A changing magnetic field gives rise to an electric field.

E A changing electric field does not give rise to a magnetic field.

F The average radial component of an electric field over a closed surface depends on the total charge of the universe.

G The electrostatic force between a proton and an electron is about 1000 times weaker than the gravitational force between them as evidenced by the fact that only very small pieces of paper are picked up by a piece of rubbed plastic.

H The speed of light in a vacuum can be calculated, using Maxwell's theory, on the basis of electrostatic and electromagnetic measurements not involving light.

Question 5 (*4 marks*) This question concerns three events $\mathscr{E}_1, \mathscr{E}_2$ and $\mathscr{E}_3$ that occur on the x-axis of an inertial frame. The spacetime coordinates of the events, in the conventional notation (ct, x) of special relativity, are given by

$$\mathscr{E}_1 = (-3a,\, a) \quad \mathscr{E}_2 = (0,\, 3a) \quad \mathscr{E}_3 = (-3a,\, 0).$$

Select from the key for Q5 the **FOUR** items that are true within the framework of special relativity.

KEY for Q5

A $\mathscr{E}_1$ is a possible cause of $\mathscr{E}_2$.

B $\mathscr{E}_1$ is a possible cause of $\mathscr{E}_3$.

C $\mathscr{E}_3$ is a possible cause of $\mathscr{E}_2$.

D There is an inertial frame in which $\mathscr{E}_1$ and $\mathscr{E}_2$ occur at the same position.

E There is an inertial frame in which $\mathscr{E}_1$ and $\mathscr{E}_2$ occur at the same time.

F There is an inertial frame in which $\mathscr{E}_1$ and $\mathscr{E}_3$ occur at the same position.

G There is an inertial frame in which $\mathscr{E}_1$ and $\mathscr{E}_3$ occur at the same time.

H There is an inertial frame in which $\mathscr{E}_2$ and $\mathscr{E}_3$ occur at the same position.

I There is an inertial frame in which $\mathscr{E}_2$ and $\mathscr{E}_3$ occur at the same time.

Question 6 (*4 marks*) The items in the key for Q6 describe various observations that might be made by observers A, B and C using inertial frames of reference. Which **FOUR** of the observations would *refute* the special theory of relativity if they really were made?

KEY for Q6

A Observers A and B have identical clocks. According to A, the interval between successive ticks of B's clock is greater by a factor γ (> 1) than that between ticks of A's own clock. According to B, the interval between successive ticks of A's clock is less by a factor $1/\gamma$ than that between ticks of B's own clock.

B Observer A is moving towards another (inertial) observer C. Observer B is moving away from observer C. A film made by observer C shows that A's clock is running fast while B's clock is running slow.

C A signal travelling at speed c according to observer C is found to be travelling at speed $v_{\rm A} > c$ by observer A and at speed $v_{\rm B} < c$ by observer B.

D A material particle travels with speed less than c in the frame of observer B, but has speed greater than c in the frame of observer A.

E A material particle travels with speed less than c in the frame of observer B, but has speed equal to c in the frame of observer A.

F Observer A sees two identical twins (of the same age, naturally) separate. One twin travels to a nearby star and then returns. Observer B sees the two twins reunited at the end of the journey, but notes that one twin has aged much more than the other.

G A metal rod observed to be of length ℓ by observer A, is found to have a lesser length ℓ' when measured by observer B.

Question 7 (*4 marks*) An observer stands on the floor of a lift that is near the surface of the Earth. A clock is fastened to the roof of the lift above the observer's head. In each of the situations (a)–(c), choose from the key for Q7 the **ONE** item that describes the behaviour of the clock according to the observer.

(a) The lift is stationary relative to the Earth. ☐

(b) The lift is in free fall. ☐

(c) The lift is accelerating upward relative to the Earth. ☐

KEY for Q7

In comparison with an identical clock held by the observer, the clock fastened to the roof of the lift is observed to run:

A fast

B slow

C at the same rate.

Question 8 (*4 marks*) Which **FOUR** of the statements in the key for Q8 are *true*? ☐☐☐☐

KEY for Q8

A The world-line of a freely-falling particle is a spacetime geodesic that *minimizes* the elapsed proper time between any pair of events on the world-line.

B The metric of a two-dimensional space with zero curvature may be written in the form

$$(\Delta l)^2 = (\Delta \rho)^2 + \rho^2 (\Delta \phi)^2.$$

C A spacetime is flat if all components of the Riemann curvature are zero at every event in that spacetime.

D The Riemann curvature throughout a region of spacetime may be non-zero even if the Ricci curvature is zero throughout that region.

E In a two-dimensional space with uniform positive curvature, the angles of a triangle sum to less than 180°.

F Einstein's field equations of general relativity relate the Ricci curvature to the energy–momentum tensor.

G The predictions of general relativity for the motion of planets in the Solar System are identical to those that follow from Newton's description of gravity in terms of an inverse square law.

H In a region of spacetime that contains no matter or radiation, the spacetime metric is necessarily that of special relativity.

Question 9 (*4 marks*) Which **FOUR** of the items in the key for Q9 are correct in the context of the general theory of relativity?

KEY for Q9

A Radio messages cannot be sent out across the photon-sphere surrounding a static black hole.

B Radio messages cannot be sent out across the event horizon surrounding a static black hole.

C A watch carried by an astronaut hovering just outside the event horizon of a black hole appears, to a distant stationary observer, to run slow.

D According to a distant observer, a body falling freely towards a black hole takes an infinite time to reach the event horizon.

E Increasing the mass of a non-rotating black hole reduces the radius of its event horizon.

F Increasing the mass of a rotating black hole reduces the radius of its event horizon.

G The only measurable properties that any black hole may possess are mass and angular momentum.

H Mini black holes, if they exist, are expected to lose energy by the process of Hawking radiation.

Question 10 (*4 marks*) The Robertson–Walker metric takes the form

$$(\Delta S)^2 = c^2(\Delta t)^2 - R^2(t)\left[\frac{(\Delta\sigma)^2}{1-k\sigma^2} + \sigma^2(\Delta\theta)^2 + \sigma^2\sin^2\theta(\Delta\phi)^2\right].$$

Select from the key for Q10 the **FOUR** statements that are *true*.

KEY for Q10

A This form of metric satisfies the cosmological principle.

B A galaxy cluster near $\sigma = 0$ occupies a privileged position in the Universe.

C The metric would make no sense for $k > 0$, since $1 - k\sigma^2$ might then vanish.

D The evolution of $R(t)$ is determined solely by the pressure.

E If $k = 0$, $R(t)$ is constant.

F If $k < 0$, $R(t)$ cannot decrease with time.

G If $k = 0$, the angles of a large triangle sum to 180°.

H If $k < 0$, the angles of a large triangle sum to more than 180°.

I If $k = 0$, then a circle with constant σ has a circumference that is $2\pi R(t)\sigma$.

J If $k < 0$, then a circle with constant σ has a circumference that is less than $2\pi R(t)\sigma$.

Question 11 (*4 marks*) In sections (*a*) and (*b*), you are to choose from the corresponding key one item for each box that best completes the statements. Write the letter for each chosen item in its box. You should not use any item from the key *more* than once.

(*a*) A [ ] describes how the intensity of electromagnetic radiation depends on frequency when it is in equilibrium with matter. The intensity of [ ] radiation appears to be highly [ ] with a [ ] of about 3 K.

KEY for Q11(*a*)

A amplitude

B temperature

C intensity

D thermal spectrum

E homogeneous

F isotropic

G frequency

H standard

I cosmic background

J constant

K mass

(*b*) According to [ ] theory, the Universe underwent a short period of very rapid expansion at an early stage in its development. A possible cause of this behaviour is the [ ] energy associated with spontaneous [ ] breaking as the Universe cools. This theory is attractive because it would resolve both the [ ] problem and the monopole problem.

KEY for Q11(*b*)

A big bang

B decoupling

C flatness

D vacuum

E inflation

F Einstein

G proton

H symmetry

I Hubble

J quantum

K Boltzmann

Question 12 (*4 marks*) A light signal from a very distant cluster of galaxies was emitted at time t_1 with wavelength λ_1. Much later it is received on Earth, at time t_2, with a longer wavelength λ_2, having suffered a cosmological redshift $z = \lambda_2/\lambda_1 - 1$.

For each of (a) to (d), select **ONE** answer from the key for Q12.

(a) What is the ratio of the large-scale density of matter at time t_1 to that at time t_2? ☐

(b) What is the ratio of the temperature of the cosmic microwave background radiation at time t_1 to that at time t_2? ☐

(c) What is the ratio of the energy density of the cosmic microwave background radiation at time t_1 to that at time t_2? ☐

(d) Assuming that the universe is not spatially flat, what is the ratio of the large-scale spatial curvature at time t_1 to that at time t_2? ☐

KEY for Q12

A 1

B z

C $1 + z$

D z^2

E $(1 + z)^2$

F z^3

G $(1 + z)^3$

H z^4

I $(1 + z)^4$

PART II (*i*) *Part II carries 52 per cent of the total marks, and you are advised to spend about* **90 minutes** on it.

(*ii*) *Attempt* **FOUR** *questions only.*

(*iii*) *All answers carry thirteen marks, and where questions are divided into more than one section (labelled (a), (b), (c), etc.), the marks allocated to each section are indicated.*

(*iv*) *Write your answer to* **each** *question from Part II in a* **separate** *answer book. You should therefore submit* **FOUR** *answer books for this part.*

Question 13 This question concerns a particle P of mass m acted on by a total force $\mathbf{F}$ that depends on the position $\mathbf{x}$ of P in an inertial frame S. The motion of P under the influence of this force conserves the value of the energy function

$$E = \frac{1}{2}m|\mathbf{v}|^2 + \frac{1}{2}k|\mathbf{x} - \mathbf{c}|^2$$

where $\mathbf{v}$ is the velocity of P in S, k is a positive constant and $\mathbf{c}$ is a constant vector.

(*a*) Use energy conservation to find the dependence of the force vector $\mathbf{F}$ on the position vector $\mathbf{x}$. (*3 marks*)

(*b*) It is found that P oscillates with a position vector $\mathbf{x}$ at time t given by

$$\mathbf{x} = \mathbf{c} + \mathbf{A}\sin(\omega t)$$

where ω is a positive constant and $\mathbf{A}$ is a constant vector. Use your answer to section (a) in Newton's second law to show that $\omega = \sqrt{k/m}$ and hence determine the period of oscillation. (*3 marks*)

(*c*) Find the velocity vector $\mathbf{v}$ of P at time S and use it to verify that the energy function of section (a) is indeed a constant. Give the value of E in terms of k and $\mathbf{A}$. (*4 marks*)

(*d*) State the two conservation laws that follow from the homogeneity and isotropy of space, taking care to define the vectorial quantities and the types of system of particles to which these laws apply. Comment on the applicability of these conservation laws to the motion of P in S. (*3 marks*)

Question 14 This question concerns a hypothetical collision of a particle with its antiparticle. Observed in an inertial frame S, each has relativistic energy $E = 45\,\text{GeV}$. Their relativistic momenta are equal in magnitude and opposite in direction. They collide and annihilate each other, producing a single photon and a new massive particle, P, with relativistic energy $E_\text{P} = 50\,\text{GeV}$.

[Note that 1 GeV is 10^9 electronvolts. In this question you will not need to convert from GeV to joules.]

(*a*) Use the conservation of relativistic energy and the fact that the photon is massless to find the magnitude of the photon's relativistic momentum. Express this magnitude of momentum in the units GeV/c. (*3 marks*)

(*b*) Use the conservation of relativistic momentum to determine the mass of P from its relativistic energy and the magnitude of its relativistic momentum. Express this mass in the units GeV/c^2. (*4 marks*)

(*c*) Show that the speed of P relative to S is $v = \frac{4}{5}c$. (*2 marks*)

(*d*) Now consider the final state of photon and particle P in a frame S′ in which P is at rest. The energy and magnitude of momentum of the photon in S′ are greater than those in S, by the Doppler shift

$$\frac{E'_\text{photon}}{E_\text{photon}} = \frac{p'_\text{photon}}{p_\text{photon}} = \sqrt{\frac{1 + v/c}{1 - v/c}}.$$

Use this result (which you need not prove) to find the *total* relativistic energy E'_total and the magnitude p'_total of the *total* relativistic momentum of the final state in frame S′ and verify that

$$(E'_\text{total})^2 - (p'_\text{total}c)^2 = (2E)^2.$$

(*4 marks*)

Question 15 This questions concerns the Schwarzschild metric

$$(\Delta\tau)^2 = \left(1 - \frac{k}{r}\right)(\Delta t)^2 - \frac{1}{c^2}\left\{\frac{(\Delta r)^2}{1 - \dfrac{k}{r}} + r^2(\Delta\theta)^2 + r^2\sin^2\theta\,(\Delta\phi)^2\right\}$$

with $k = 2GM/c^2$. It describes the spacetime exterior to an isolated spherically symmetric star S of mass M.

In this question you may use the approximations

$$\sqrt{1-x} \approx 1 - \frac{1}{2}x$$

and

$$1/(1-x) \approx 1 + x,$$

whenever x is an expression that has c^2 in its denominator.

(*a*) Laser A is maintained at fixed Schwarzschild coordinates ($r = R_1$, $\theta = \pi/2$, $\phi = 0$) with $R_1 \gg k$. It transmits radially outwards at a frequency f_0, as recorded by a local observer. Observer O has fixed Schwarzschild coordinates ($r = R_0$, $\theta = \pi/2$, $\phi = 0$), with $R_0 \gg R_1$. O receives A's signal and determines its frequency to be f_1. Show that

$$\frac{f_0 - f_1}{f_0} \approx \frac{GM}{R_1 c^2}$$

taking care to explain how you use the metric. (*4 marks*)

(*b*) An identical laser, B, is aboard a spaceship that orbits S at constant speed in a circle with $r = R_2 \ll R_0$ and $\theta = \pi/2$. According to O, the speed of B is $v \ll c$. Laser B also transmits radially outwards. O receives B's signal (once per orbit) and determines its frequency to be f_2. Show that

$$\frac{f_0 - f_2}{f_0} \approx \frac{GM}{R_2 c^2} + \frac{v^2}{2c^2}$$

taking care to explain how you use the metric. (*4 marks*)

(*c*) Suppose that the spaceship carrying B had no motors and hence was describing a geodesic. Use Newton's laws of motion under gravity (which are adequate in this situation) to relate the speed v and radius R_2 of B's circular orbit and hence show that O receives the same frequency from B as from A when $R_2 = 3R_1/2$. (*3 marks*)

(*d*) Consider the argument that 'O cannot receive the same frequency from B as from A, since B describes a geodesic, but A does not'. Write a concise sentence that exposes and corrects the flaw in this argument. (*2 marks*)

Question 16 This question concerns the evolution of a hypothetical homogeneous universe, for which Einstein's field equations lead to:

$$\left(\frac{1}{R(t)}\frac{\mathrm{d}R(t)}{\mathrm{d}t}\right)^2 + \frac{kc^2}{R^2(t)} = \frac{8\pi G}{3c^2}\rho(t)$$

where $R(t)$ and $\rho(t)$ are the scale factor and energy density at universal time t, the constant k is the spatial curvature parameter, and

$$\frac{8\pi G}{3c^2} = 6.2 \times 10^{-27}\,\mathrm{m\,kg^{-1}}.$$

(*a*) Suppose that at some particular time, t_0, the Hubble parameter has the value $2.0 \times 10^{-18}\,\mathrm{s^{-1}}$ and the energy density has the value $6.0 \times 10^{-10}\,\mathrm{J\,m^{-3}}$. Determine the sign of the spatial curvature parameter and give two properties of large geometrical figures that follow from your answer. (*4 marks*)

(*b*) State how the contributions of matter and radiation to the energy density depend on the scale factor, giving brief explanations for the powers of R in your answers. (*4 marks*)

(*c*) Show that at times $t \ll t_0$ the relationship $\mathrm{d}R/\mathrm{d}t \propto 1/R$ results from the domination of radiation and hence that $R \propto t^{1/2}$. (*3 marks*)

(*d*) Show that at times $t \gg t_0$ the relationship $R \propto t$ is obtained for this particular universe. (*2 marks*)

Question 17 Consider the proposition that 'the prediction of the event horizon of a black hole owes much to delicate measurements made within the Solar System.' Support this argument by explaining how the general theory of relativity

(i) agrees largely, but not completely, with Newtonian predictions for the motions of planets, yet does not use the concept of force;

(ii) leads to successful predictions about the motion, frequency shifts and time delays of light signals in the Solar System;

(iii) uses the same metric as for the Solar System to predict radically new effects in the spacetime produced by the collapse of a spherically symmetric star.

(*13 marks*)

Aim to use fewer than about 300 words. Marks will be given primarily for the relevance and clarity of your explanations.

Question 18 Explain how observations of the cosmic microwave background radiation have been interpreted as supporting the following views.

(i) The Universe was much denser and hotter in the past.

(ii) Our motion relative to a frame of large-scale isotropy of the Universe is now known to fair precision.

(iii) The origin of the high degree of isotropy of the early Universe is rather puzzling.

(*13 marks*)

Aim to use fewer than about 300 words. Marks will be given primarily for the relevance and clarity of your explanations.

[END OF QUESTION PAPER]