Q.2–4 Consider the evolution of the spin state of an electron of mass m and charge $-e$ in a magnetic field of magnitude B directed along the x-axis. The hamiltonian matrix is

$$\mathbf{H} = \tfrac{1}{2}\hbar\omega \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where $\omega \simeq eB/m$.

(i) Show that

$$\mathbf{e}_+ = \frac{1}{\sqrt{2}}[1 \quad 1]^T \quad \text{and} \quad \mathbf{e}_- = \frac{1}{\sqrt{2}}[-1 \quad 1]^T$$

are orthonormal spinors.

(ii) Verify that $\mathbf{e}_+$ and $\mathbf{e}_-$ are eigenvectors of $\mathbf{H}$ and find the corresponding eigenvalues E_+ and E_-.

The spin state of the electron at any time t is represented by the spinor

$$\mathbf{a}_t = a_+ \exp(-\mathrm{i}E_+t/\hbar)\mathbf{e}_+ + a_- \exp(-\mathrm{i}E_-t/\hbar)\mathbf{e}_-,$$

where a_+ and a_- are constant amplitudes. The initial state is represented by the spinor

$$\mathbf{a}_0 = [1 \quad 0]^T$$

which is an eigenstate of

$$\mathbf{S}_z = \tfrac{1}{2}\hbar \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with eigenvalue $\frac{1}{2}\hbar$.

(iii) Evaluate the amplitudes a_+ and a_-. Hence show that

$$\mathbf{a}_t = [\cos \tfrac{1}{2}\omega t \quad -\mathrm{i}\sin \tfrac{1}{2}\omega t]^T.$$

(iv) Calculate the probability that a measurement of S_z at time t will yield the result $\frac{1}{2}\hbar$.

(v) Using the answer to part (iv), or otherwise, show that the expectation value of a measurement of S_z at time t is $\frac{1}{2}\hbar\cos\omega t$.

You may find the following identities useful:

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta, \quad e^{-\mathrm{i}\theta} = \cos\theta - \mathrm{i}\sin\theta, \quad \cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2).$$
