Q.2–3 This question concerns the isotropic three-dimensional harmonic oscillator whose hamiltonian is

$$\hat{\mathrm{H}} = -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \tfrac{1}{2}M\omega^2\left(x^2 + y^2 + z^2\right).$$

The energy eigenfunctions of the one-dimensional harmonic oscillator are denoted by $\phi_n(x)$, corresponding to energy eigenvalues

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega \qquad (n = 0, 1, 2, \ldots).$$

You may assume that the eigenfunctions for the two lowest states of the one-dimensional harmonic oscillator are

$$\phi_0(x) = \left(\frac{1}{a\sqrt{\pi}}\right)^{1/2} \exp\left(-\frac{x^2}{2a^2}\right)$$

and

$$\phi_1(x) = \left(\frac{2}{a^3\sqrt{\pi}}\right)^{1/2} x \exp\left(-\frac{x^2}{2a^2}\right),$$

where $a = (\hbar/M\omega)^{1/2}$.

(i) By expressing the hamiltonian $\hat{\mathrm{H}}$ as $\hat{\mathrm{H}}_x + \hat{\mathrm{H}}_y + \hat{\mathrm{H}}_z$, where $\hat{\mathrm{H}}_x$, $\hat{\mathrm{H}}_y$ and $\hat{\mathrm{H}}_z$ are the hamiltonians of the one-dimensional harmonic oscillators in x, y and z, respectively, show that

$$\psi_{l,m,n}(x, y, z) = \phi_l(x)\,\phi_m(y)\,\phi_n(z)$$

is an energy eigenfunction of the three-dimensional harmonic oscillator and find the corresponding energy eigenvalue.

(ii) Write down the eigenfunction for the ground state of the three-dimensional harmonic oscillator.

(iii) Show that the ground state of the three-dimensional harmonic oscillator is an eigenstate of the angular momentum operator

$$\hat{\mathrm{L}}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

and obtain the corresponding eigenvalue.

(iv) Find the degeneracies of the first and second excited states of the three-dimensional harmonic oscillator. Specify the corresponding eigenfunctions in terms of the one-dimensional harmonic oscillator eigenfunctions ϕ_n.

(v) Show that

$$\psi(x, y, z) = \frac{1}{\sqrt{2}}\left(\phi_1(x)\,\phi_0(y)\,\phi_1(z) - i\phi_0(x)\,\phi_1(y)\,\phi_1(z)\right)$$

is an eigenstate of $\hat{\mathrm{L}}_z$ and obtain the corresponding eigenvalue.