Q.2–5 This question concerns an electron in a state represented by the spinor of functions

$$\Psi = \frac{u(r)}{r}\begin{bmatrix}\sqrt{\frac{5}{7}} & Y_{3,1}(\theta,\phi)\\ \sqrt{\frac{2}{7}} & Y_{3,2}(\theta,\phi)\end{bmatrix},$$

where $u(r)$ is a radial function normalized by

$$\int_0^\infty |u(r)|^2\,dr = 1.$$

You may assume (but should not attempt to prove) that Ψ is an eigenvector of $\hat{J}^2$.

(i) Show that Ψ is an eigenvector of $\hat{J}_z = \hat{L}_z + \hat{S}_z$ and find the eigenvalue m_j.

The spinor representation of $\hat{S}_z$ is

$$\hat{S}_z = \tfrac{1}{2}\hbar\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

(ii) Calculate $\langle S_z\rangle$ for the state represented by Ψ.

(iii) Using the answers to parts (i) and (ii), or otherwise, calculate $\langle L_z\rangle$ for the state represented by Ψ.

(iv) Calculate the Landé g factor for this state, using the definition

$$g = \frac{\langle L_z\rangle + 2\langle S_z\rangle}{\hbar m_j}.$$

(v) For eigenstates of $\hat{J}^2, \hat{L}^2, \hat{S}^2$ and $\hat{J}_z$, it can be shown that

$$g = \frac{j+\frac{1}{2}}{l+\frac{1}{2}}$$

when $s = \frac{1}{2}$. Use this result to find the eigenvalue of $\hat{J}^2$ for the state represented by Ψ.

Q.2–6

(i) Describe in a few sentences the main ideas of the *variational method* for estimating the ground state energy of a particle in a potential well. State how the estimate is related to the true ground state energy.

(ii) Using trial wave functions of the form

$$\psi_t(x) = \exp(-\alpha x^2)$$

where α is a positive parameter, show that the expectation value of the energy for the one-dimensional harmonic oscillator, whose potential energy function is $V(x) = \frac{1}{2}m\omega^2x^2$, according to the variational method, is

$$\langle E\rangle = \frac{\hbar^2\alpha}{2m} + \frac{m\omega^2}{8\alpha}.$$

(iii) Using the result of part (ii), find the best estimate of the ground state energy of the harmonic oscillator which can be found using trial wave functions of the above form.

(iv) State and briefly explain how your estimate in part (iii) is related to the true ground state energy of the linear harmonic oscillator.

You may find the following integrals useful:

$$\int_{-\infty}^{\infty}\exp(-ax^2)dx = \sqrt{\frac{\pi}{a}},\qquad \int_{-\infty}^{\infty}x^2\exp(-ax^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}.$$
