(d) For $x < 0$,

$$J = -\frac{i\hbar}{2m}\{(A^* e^{-ikx} + B^* e^{ikx})(ikAe^{ikx} - ikBe^{-ikx})$$
$$- (Ae^{ikx} + Be^{-ikx})(-ikA^* e^{-ikx} + ikB^* e^{ikx})\}$$
$$= -\frac{i\hbar}{2m}\{2A^* Aik - 2B^* Bik\}$$
$$= \frac{\hbar k}{m}(|A|^2 - |B|^2).$$

(e) For $x > 0$, the result $J = 0$ shows that no particles are transmitted. This requires $R = 1$ and hence $|A|^2 = |B|^2$, giving $J = 0$ for $x < 0$, i.e. probability currents for incident and reflected beams cancel.

Thus, results (c) and (d) are consistent with the result $R = 1$.

Q.2–3 (i)

$$\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} |R_{n,l}(r)P_{l,m}(\theta)\Phi_m(\phi)|^2 r^2 \sin\theta \, dr\, d\theta\, d\phi$$
$$= \left(\int_0^{\infty} |R_{n,l}(r)|^2 r^2 \, dr\right)\left(\int_0^{\pi} |P_{l,m}(\theta)|^2 \sin\theta \, d\theta\right)$$
$$\times \left(\int_0^{2\pi} |\Phi_m(\phi)|^2 \, d\phi\right).$$

(ii) $(\psi_{2,1,0}, \widehat{\mathbf{V}}\psi_{2,1,0}) \propto \int_0^{\pi} \cos^3\theta \sin\theta \, d\theta$

$$= -\left[\frac{\cos^4\theta}{4}\right]_0^{\pi} = 0$$

$$(\psi_{2,1,\pm1}, \widehat{\mathbf{V}}\psi_{2,1,\pm1}) \propto \int_0^{\infty} \cos\theta \sin^3\theta \, d\theta$$
$$= \left[\frac{\sin^4\theta}{4}\right]_0^{\pi} = 0$$

(iii)

$$(\psi_{2,1,0}, \widehat{\mathbf{V}}'\psi_{2,1,0}) = \int_0^{\infty} |f(r)|^2 A e^{-\lambda' r} r^4 \, dr$$
$$\times \int_0^{\pi} \cos^4\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi$$
$$= 2\pi A \int_0^{\infty} |f(r)|^2 e^{-\lambda' r} r^4 \, dr \left[-\frac{\cos^5\theta}{5}\right]_0^{\pi}$$
$$= \tfrac{4}{5}\pi A \int_0^{\infty} |f(r)|^2 e^{-\lambda' r} r^4 \, dr$$

$$(\psi_{2,1,\pm1}, \widehat{\mathbf{V}}'\psi_{2,1,\pm1}) = \tfrac{1}{2}\int_0^{\infty} |f(r)|^2 A e^{-\lambda' r} r^4 \, dr$$
$$\times \int_0^{\pi} \sin^3\theta \cos^2\theta \, d\theta \int_0^{2\pi} d\phi$$
$$= \tfrac{4}{15}\pi A \times \text{ (integral over } r)$$

The splitting produced is of magnitude

$$\tfrac{8}{15}\pi A \times \text{ (integral over } r).$$

Q.2–4 (i) $\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}Cx^2\right)\psi(x) = E\psi(x).$

Substituting the given wave function for ψ yields

$$-\frac{\hbar^2}{2m}[-2\gamma(1 - 2\gamma x^2)] + \tfrac{1}{2}Cx^2 = E_n$$

coefficients of x^2: $-\frac{\hbar^2}{2m}4\gamma^2 + \frac{1}{2}C = 0$

$$\therefore \gamma = \frac{1}{2\hbar}(Cm)^{1/2}$$

constant terms: $-\frac{\hbar^2}{2m}(-2\gamma) = E_n$

$$\therefore E_n = \frac{\hbar^2}{m}\frac{1}{2\hbar}(Cm)^{1/2} = \frac{1}{2}\hbar\left(\frac{C}{m}\right)^{1/2}.$$

(ii) Even.

$$\therefore \langle x\rangle = \left(\frac{2\gamma}{\pi}\right)^{1/2}\int_{-\infty}^{\infty} x\exp(-2\gamma x^2)\, dx = 0$$

and

$$\langle p_x\rangle = \left(\frac{2\gamma}{\pi}\right)^{1/2}\cdot\left(\frac{\hbar}{i}\right)\int_{-\infty}^{\infty}\exp(-\gamma x^2)\frac{\partial}{\partial x}(\exp(-\gamma x^2))\, dx$$
$$\propto \int_{-\infty}^{\infty} x\exp(-\gamma x^2)\, dx = 0.$$

(iii)

$$\langle p_x^2\rangle = -\left(\frac{2\gamma}{\pi}\right)^{1/2}\hbar^2\int_{-\infty}^{\infty}\exp(-\gamma x^2)\frac{\partial^2}{\partial x^2}(\exp(-\gamma x^2))\, dx$$
$$= \left(\frac{2\gamma}{\pi}\right)^{1/2}\hbar^2\int_{-\infty}^{\infty}(2\gamma\exp(-2\gamma x^2) - 4\gamma^2 x^2\exp(-2\gamma x^2))\, dx$$
$$= \left(\frac{2\gamma}{\pi}\right)^{1/2}\hbar^2(2\gamma)\int_{-\infty}^{\infty}\exp(-2\gamma x^2)\, dx - 4\gamma^2\hbar^2\langle x^2\rangle$$
$$= \hbar(Cm)^{1/2} - \frac{1}{2}(Cm)^{1/2} = \frac{1}{2}\hbar(Cm)^{1/2}.$$

(iv)

$$\Delta(x) = (\langle x^2\rangle - \langle x\rangle^2)^{1/2} = (\hbar/2)^{1/2}(Cm)^{-1/4}$$
$$\Delta(p_x) = (\langle p_x^2\rangle - \langle p_x\rangle^2)^{1/2} = (\hbar/2)^{1/2}(Cm)^{1/4}.$$

$\therefore \Delta(p_x)\Delta(x) = \hbar/2$, which is consistent with the uncertainty principle $\Delta(p_x)\Delta(x) \geq \hbar/2$.

Q.2–5 (i) Definition: $\mathrm{Sp}(\widehat{\mathbf{A}})$ is the set $\{\lambda\}$ of numbers λ for each of which there exists a sequence $\{\mathbf{\Phi}_n^{\lambda}\}$ of normalized vectors such that the number sequence $\{c_n^{\lambda}\}$ given by $c_n^{\lambda} = (\boldsymbol{\eta}_n^{\lambda}, \boldsymbol{\eta}_n^{\lambda})$ with $\boldsymbol{\eta}_n^{\lambda} = (\widehat{\mathbf{A}} - \lambda\widehat{\mathbf{1}})\mathbf{\Phi}_n^{\lambda}$ converges to zero as $n \to \infty$.

$\mathrm{Sp}(\widehat{\mathbf{A}})$ is needed to generalize the measurement postulates to cope with $\widehat{\mathrm{x}}$, which has neither eigenvalues nor eigenvectors, and operators like $\widehat{\mathrm{p}}$ and $\widehat{\mathrm{H}}_{\mathrm{fsw}}$ that do not have a complete set of orthonormal eigenvectors.

(ii) Consider a sequence of normalized vectors $\{\mathbf{\Phi}_n^{E}\}$ with

$$\mathbf{\Phi}_n^{E} = \sum_{m=0}^{\infty} b_{nm}\psi_m$$

where the ψ_m are the complete set of simple harmonic oscillator wave functions.