MST121/C

First Level Course Examination 1999
Using Mathematics

Tuesday, 12 October, 1999 10.00 am – 1.00 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I you should attempt as many questions as you can, writing your answers *in the spaces provided* inside this examination paper. You should attempt not more than **TWO** questions in Part II. Your answers to this part should be written in the answer book provided.

80% of the available marks are assigned to Part I and 20% to Part II. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part I before starting Part II.

Graph paper is available from the invigilator, if you feel it would assist you in answering questions.

At the end of the examination

Check that you have completed the grid below, and have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

Examination No.								
Personal Identifier								

Calculator

Indicate in the box below the make and model number of the calculator which you used in this examination.

Attach this examination paper to the FRONT of the answer book(s) in which you have answered questions from Part II. Put your signed desk record on top, and fix them all together with the fastener provided.

PART I

Instructions

(i) *You should attempt as many questions as you can in this part of the examination.*

(ii) *Part I carries 80% of the available examination marks. Each question indicates how many of these marks are allocated to it.*

(iii) *You should, as far as possible, record your answers to each question in this part in the space provided on the question paper. You are strongly advised to show all your working, including any rough working. If you need extra space then you may continue your working in a separate answer book. If you do this, make sure that your work is clearly labelled.*

Question 1 – 4 marks

The formula

$$x_n = 3 - 0.1n \quad (n = 0, 1, 2, \ldots)$$

generates a sequence.

(a) Complete the following table to give the first 4 terms of the sequence. [1]

x_n	value of x_n
x_0	
x_1	
x_2	
x_3	

(b) State what kind of sequence this is and express it as a recurrence system. [2]

(c) Suppose x_n represents the distance of a particle from a fixed point, O, at n seconds after a specific instant of time. State whether the particle will ever reach O and, if so, after how long. [1]

Question 2 – 4 marks

The first four terms of a sequence, in order, are:

$$6, 2, \tfrac{2}{3}, \tfrac{2}{9}$$

Assuming that the sequence continues indefinitely and that terms continue to follow the above pattern:

(a) write down the next two terms of the sequence; [1]

(b) state what kind of sequence this is and give its closed form; [2]

(c) state what happens to this sequence in the long run. [1]

Question 3 – 6 marks

The equation

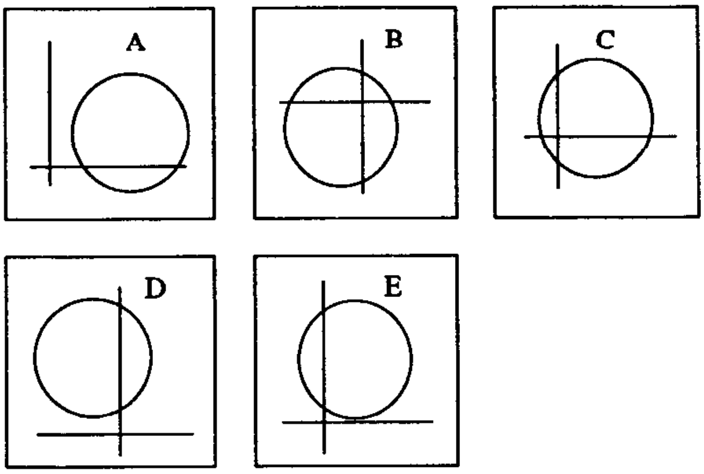
$$x^2 + y^2 - 6x - 2y - 6 = 0$$

represents a circle.

(a) Express the equation in standard form (completed square form) and hence find the centre and radius of the circle. [3]

(b) Find the distance of the centre from the origin. [1]

(c) One of the following diagrams shows a sketch of the circle. State which diagram this is, and give a reason for your choice. [2]



Question 4 – 6 marks

A bank publishes literature explaining the operation of its saving schemes. In one example the following formula is given for the accumulated value V after n years of an investment of £2000.

$$V = 2000(1 + 0.04)^n$$

(a) What is the annual interest rate assumed by this formula? [1]

(b) If we write $f(n)$ to represent the value of the investment after n years, write down a formal definition of the continuous function f which approximates this value. [2]

(c) Determine f^{-1}, the inverse of the function f, and hence calculate the number of years required for the value of the investment to exceed £3000. [3]

Question 5 – 4 marks

(a) Complete the table on the right below in order to trace the given algorithm in which the input $x \in \mathbf{R}$. [3]

	memone	*memtwo*	*memthree*
memone := x			
memtwo := *memone* −1			
memtwo := *memtwo*^{2}			
memthree := *memone*^{2}			
memthree := *memthree* −4			
memtwo := *memtwo*/*memthree*			

The calculated value is in *memtwo*.

(b) State all values of x for which the algorithm will fail. [1]

Question 6 – 6 marks

In this question $n \in \mathbb{N}$, $a = 1.25$ and $k = -0.5$.

(a) (i) Evaluate $\sum_{r=1}^{n}(a + rk)$ for $n = 4$. [1]

(ii) Describe briefly, giving reasons, what happens to this sum when n becomes very large. [2]

(b) (i) Use the formula

$$\sum_{r=1}^{n} ak^r = ak\frac{(1-k^n)}{(1-k)}$$

to evaluate the sum when $n = 4$. [1]

(ii) Describe briefly, giving reasons, what happens to this sum when n becomes very large. [2]

Question 7 – 4 marks

(a) A population is modelled by the logistic recurrence equation

$$P_{i+1} - P_i = 2.1P_i\left(1 - \frac{P_i}{3000}\right), \quad i = 0, 1, 2, 3, \ldots \quad \text{with } 0 < P_0 < 3000.$$

Describe what will happen to this population in the long-term (that is for large values of i) giving brief reasons for your answer. [1]

(b) Another population is modelled by the recurrence equation

$$P_{i+1} = 1.7P_i - 3.5 \times 10^{-4}P_i^2, \quad i = 0, 1, 2, 3, \ldots,$$

where P_i is measured in millions. If this population is in equilibrium (i.e. $P_{i+1} = P_i = E$) find the value of the equilibrium population E. [3]

Question 8 – 6 marks

(a) For the matrices $\mathbf{P} = \begin{pmatrix} 0 & 6 \\ -1 & -3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$, calculate each of the following

(i) $\mathbf{P} + \mathbf{Q}$ (ii) $\mathbf{QP}$ [2]

(b) The pair of equations

$$2u_1 - 5u_2 = -9$$
$$-u_1 + 4u_2 = 6$$

is to be expressed in vector-matrix form $\mathbf{Mu} = \mathbf{k}$, where $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

(i) Write down the matrix $\mathbf{M}$ and the vector $\mathbf{k}$. [1]

(ii) Write down the inverse of $\mathbf{M}$, and hence solve the equations. [3]

Question 9 – 5 marks

The equation

$$h = 35 + 30t - 5t^2$$

is to be used as a model for the motion of an object catapulted vertically upwards with a particular initial speed and height. Here h metres is the height above ground level and t seconds is the time since the moment of projection.

(a) Express the velocity $v\,\mathrm{m\,s^{-1}}$ of the object in terms of t. [1]

(b) What is the maximum height above ground level which is attained by the object? [2]

(c) Assuming that it meets no obstacle beforehand, at what time does the object reach ground level? [2]

Question 10 – 6 marks

(a) Differentiate the function

$$f(t) = 3\exp(-t) - 4t^3 \quad (t \in \mathbf{R}),$$

identifying any general rules which you use. [3]

(b) Find the indefinite integral of the function

$$g(x) = \frac{2}{x} + 5\sin(5x) \quad (x > 0),$$

identifying any general rules which you use. [3]

Question 11 – 4 marks

(a) Find the x-coordinates of the two points at which the graph of the function

$$f(x) = (x-2)(1-e^x)$$

crosses the x-axis, and explain why the graph is above the x-axis between these two points. [2]

(b) According to Mathcad, an integral of $f(x)$ is given by

$$\tfrac{1}{2}x^2 - 2x + (3-x)e^x.$$

Use this expression to find the area enclosed by the graph of $f(x)$ and by the x-axis. [2]

Question 12 – 5 marks

The temperature in an apartment room is 35°C at the moment when the air conditioning system is first switched on. After a long time the room temperature settles at 20°C. The temperature θ°C at time t hours after the heating is switched on is to be modelled by a function of the form

$$\theta = A + B\exp(-0.43t).$$

(a) Sketch the graph of this temperature function, referring to the specific temperatures given above. [1]

(b) Find the appropriate values of the parameters A and B. [2]

(c) According to this model, what is the time at which the temperature in the room reaches 25°C? [2]

Question 13 – 8 marks

A regular tetrahedral die has four faces labelled 1, 2, 3 and 4. When it is rolled, it is equally likely to land on each of the four faces. The score obtained is the number on the face on which it lands. Two tetrahedral dice are rolled together.

(a) Find the probability that the scores on the two dice add up to 4. [2]

(b) Write down the probability that the scores on the two dice are the same (a 'double'). [1]

The two dice are rolled together repeatedly.

(c) Find the probability that none of the first 8 rolls results in a double. [2]

(d) Find the probability that there is at least one double in the first 8 rolls of the two dice. [1]

(e) The two dice are rolled until a double is obtained. How many times would you expect the two dice to be rolled – that is, what is the mean number of rolls of two tetrahedral dice required to obtain a double? [2]

Question 14 - 6 marks

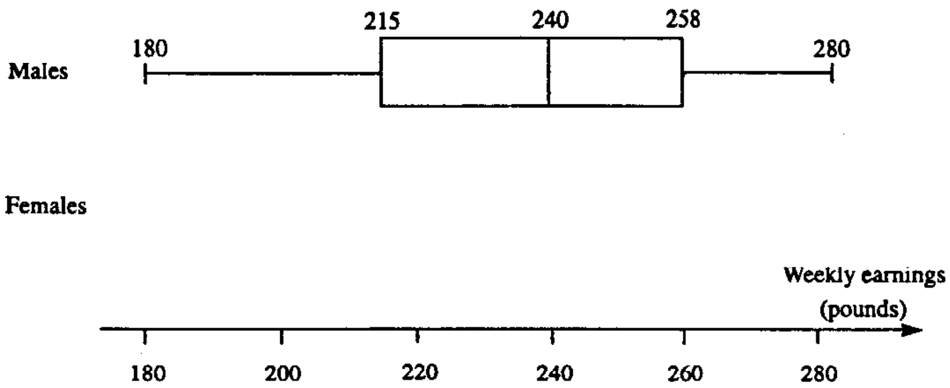
The gross weekly earnings in pounds of a sample of 8 female trainee managers in a large company are given below.

192 220 228 239 245 245 257 262

(a) Find the median, the lower quartile, the upper quartile, the range and the interquartile range for this sample of earnings and write your answers in the table below. [3]

Median	
Lower quartile	
Upper quartile	
Range	
Interquartile range	

(b) The boxplot below represents the gross weekly earnings in pounds of a sample of male trainee managers with the same company. In the space provided in the diagram, draw a boxplot to represent the gross weekly earnings of the sample of female trainees given above. [1]



(c) Use the boxplots to compare the earnings of the male trainee managers and the female trainee managers. Your comparison should include comments on the relative size of the earnings of the men and the women and on the variation in the earnings of the men and the women. [2]

Question 15 - 3 marks

In a certain country, the variation in the birthweights of boys may be modelled by a distribution with mean 3430 grams and standard deviation 487 grams.

Calculate a range of values within which the mean birthweight (in grams) of approximately 95% of samples of 400 boys will lie. [3]

Question 16 - 3 marks

In a study of physical measurements within families, data are collected for a large number of families on the heights in centimetres of adult men and their adult sisters. The data are used to calculate the equations of two least squares fit lines. The equation of the first, which is labelled (1) below, uses brother's height as the explanatory variable; and the equation of the second, which is labelled (2) below, uses sister's height as the explanatory variable. In each case, x is the explanatory variable.

(1) $y = 69 + 0.57x$

(2) $y = 83 + 0.54x$

(a) According to the appropriate model, on average how much taller than their sisters are the brothers of women who are 160 centimetres tall? [2]

(b) Imran is 10 centimetres taller than his friend John. According to the appropriate model, what is the predicted difference in height between Imran's sister and John's sister? [1]

PART II

Instructions

(i) *You should attempt not more than* **TWO** *questions from this part of the examination.*

(ii) *Each question in this part carries 10% of the marks.*

(iii) *You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.*

(iv) *Show all your working.*

Question 17

The position of a point on a curve in the (x, y)-plane is given by the parametric coordinates

$$(2t - 1, 4t^2 - 8t) \quad (t \subset \mathbb{R}).$$

(a) (i) When $t = 0$ the point is at A and when $t = 3$ the point is at B. Find the coordinates of A and B. [1]

(ii) Eliminate the parameter t and find the equation of the curve in the form $y = f(x)$. [2]

(b) (i) By 'completing the square' on the expression $f(x)$, or otherwise, find the minimum value of y and the value of x at which this minimum value occurs. [2]

(ii) Find the coordinates of the points where the curve cuts the axes and hence sketch the curve. [3]

(c) Find the equation of the chord AB. [2]

Question 18

This question asks you to interpret the information given on the Mathcad screen opposite set up to investigate the behaviour over time of a matrix model of the population of a given country. The population is divided into two sub-populations referred to as 'young' (those people who are younger than 65 years) and 'elderly' (those people who are aged 65 years or more). Both sub-populations are counted in millions.

(a) (i) Write down the initial values of Y_0 and E_0, and find the proportion of elderly people in this population when $i = 0$. [1]

(ii) Write down the equations for Y_{n+1} and E_{n+1} in terms of Y_n and E_n which have been used in expressing the matrix model. Describe briefly the meaning of each of the elements of the matrix. [3]

(b) Sketch graphs to show how you would expect the 'ratio of total increase' and the 'proportion of elderly' to behave if the model were run for 70 years. [2]

(c) Describe what the model predicts will happen in the long term to:

(i) the size of the total population; [2]

(ii) the proportion of elderly people in the population. [2]

Population equations for young and elderly

Number of years $N := 50$

Calculation range $i := 0, 1..N-1$ Matrix $M := \begin{pmatrix} 0.9991 & 0 \\ 0.0143 & 0.9325 \end{pmatrix}$

Access and graph range $k := 0, 1..N$

Recurrence system $P^{<0>} := \begin{pmatrix} 48.42 \\ 8.99 \end{pmatrix}$ and $P^{<i+1>} := M \cdot P^{<i>}$

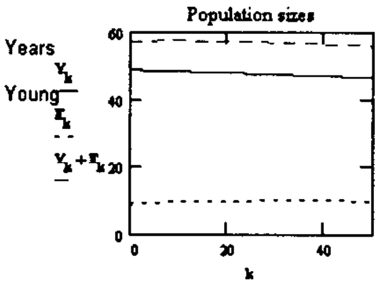
Extract the two sub-populations from the population vector -

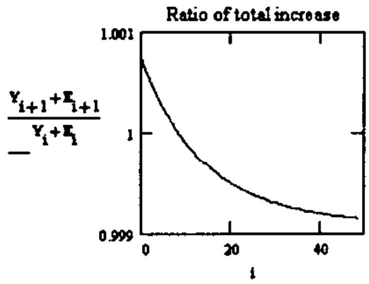
Young (aged < 65) $Y_k := \left(P^{<k>}\right)_0$ Elderly (aged ≥ 65) $E_k := \left(P^{<k>}\right)_1$

Long-term behaviour of the populations

$N = 50$ Total $Y_N + E_N = 56.185$ $\frac{Y_N + E_N}{Y_{N-1} + E_{N-1}} = 0.999$

$Y_N = 46.288$ Elderly $E_N = 9.896$



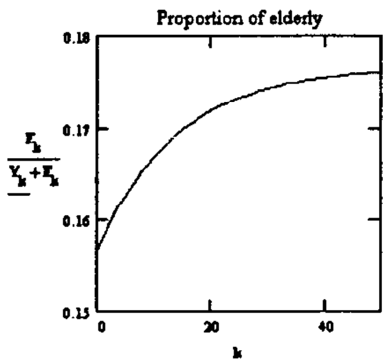


Proportion of elderly in the total population

Number of years $N = 50$

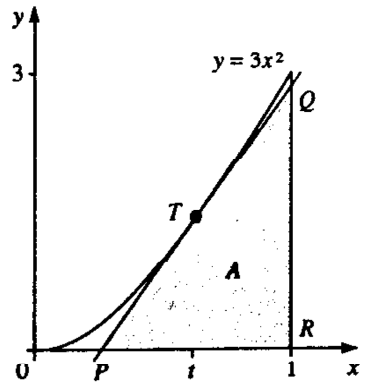
Proportion of young $\frac{Y_N}{Y_N + E_N} = 0.824$

Proportion of elderly $\frac{E_N}{Y_N + E_N} = 0.176$



Question 19

The figure below shows part of the graph of the function $f(x) = 3x^2$.



(a) Find the area bounded by the graph of $f(x) = 3x^2$, the x-axis and the line $x = 1$. [NB: This is *not* the area shown shaded and labelled as A in the figure.] [2]

(b) The point T on the graph has coordinates $(t, 3t^2)$. The tangent line drawn on the figure passes through T and has slope equal to $f'(t)$. Verify that this line has equation

$$y = (6t)x - 3t^2.$$ [2]

(c) This line meets the x-axis at P and meets the line $x = 1$ at Q. The point R has coordinates $(1, 0)$. Show that the triangle PQR has area

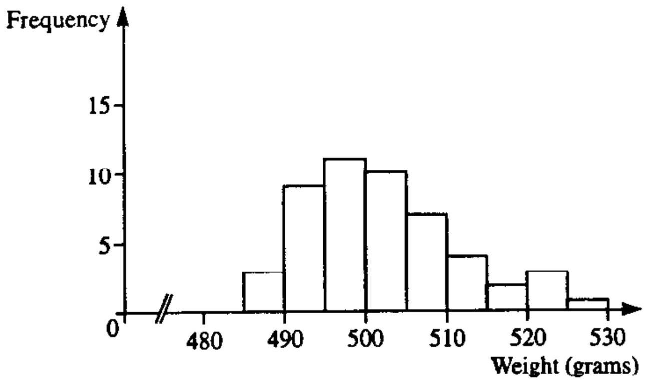
$$A = \tfrac{3}{4}t(2-t)^2.$$ [3]

(d) As the point T is moved along the graph of $f(x) = 3x^2$ from the origin to $(1, 3)$, the area A varies. Find the maximum value of this area, and check that this is less than the area found in part (a). [3]

Question 20

A food manufacturer has just installed a new machine for packing bags of sugar; the bags are labelled as containing 500 grams.

(a) The frequency diagram below represents the weights in grams of a sample of 50 bags taken from the first hour's production.



(i) Give one reason why it would not be appropriate to use a normal distribution to model the variation in the weights of bags produced on the machine. [1]

(ii) Sketch a curve which you think might provide a reasonable model for the variation in the weights of bags produced on the machine. You should include a horizontal scale on your sketch, but there is no need to include a vertical scale. [1]

(iii) With the assistance of your sketch, explain how you might use your model to estimate the proportion of all the bags of sugar produced on the machine that weigh less than 500 grams. [2]

(b) By law, the mean weight of bags must be at least 500 grams (the nominal weight). The mean weight of the sample of 50 bags is 502.4 grams and the sample standard deviation is 9.79 grams.

(i) Calculate a 95% confidence interval for the mean weight of bags of sugar produced on this machine. [2]

(ii) What can the manufacturer conclude from this confidence interval about the mean weight of bags of sugar produced on this machine? [1]

(iii) Before deciding whether or not to adjust the machine (and so change the mean), the production manager decides to take another sample of bags and calculate a second confidence interval for the mean weight using this sample. He wants the width of this confidence interval to be half the width of the confidence interval in (b)(i).

(1) Approximately what size sample should he take? Briefly justify your answer.

(2) Will the width of the confidence interval calculated from a sample of this size be exactly one half of the width of the confidence interval in (b)(i)? Explain your answer. [3]

[END OF QUESTION PAPER]