The Open University

MST121/C

First Level Course Examination 1998

Using Mathematics

Tuesday 13 October, 1998 10.00 am – 1.00 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I you should attempt as many questions as you can, writing your answers *in the spaces provided* inside this examination paper. You should attempt not more than **TWO** questions in Part II. Your answers to this part should be written in the answer book provided.

80% of the available marks are assigned to Part I and 20% to Part II. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part I before starting Part II.

Graph paper is available from the invigilator, if you feel it would assist you in answering questions.

At the end of the examination

Check that you have completed the grid below, and have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

Examination No.								
Personal Identifier								

Calculator

Indicate in the box below the make and model number of the calculator which you used in this examination.

Attach this examination paper to the FRONT of the answer book(s) in which you have answered questions from Part II. Put your signed desk record on top, and fix them all together with the fastener provided.

NB: To aid downloading I have eliminated many of the spaces provided for writing answers.

PART I

Instructions

(i) *You should attempt as many questions as you can in this part of the examination.*

(ii) *Part I carries 80% of the available examination marks. Each question indicates how many of these marks are allocated to it.*

(iii) *You should, as far as possible, record your answers to each question in this part in the space provided on the question paper. You are strongly advised to show all your working, including any rough working. If you need extra space then you may continue your working in a separate answer book. If you do this, make sure that your work is clearly labelled.*

Question 1 – 3 marks

The formula

$$x_n = 7 - 0.2(n-1) \qquad (n = 1, 2, \ldots)$$

generates a sequence.

(a) Write down the first 4 terms of the sequence. [1]

(b) State what kind of sequence this is and express it as a recurrence system. [2]

Question 2 – 3 marks

The recurrence system

$$x_0 = 1, \quad x_{n+1} = \frac{x_n}{2} \qquad (n = 0, 1, 2, \ldots)$$

generates a sequence.

(a) Write down the first 4 terms of the sequence. [1]

(b) State what kind of sequence this is and give its closed form. [2]

Question 3 – 6 marks

An object moves in the (x, y)-plane so that its location is given by the parametric coordinates $(3t - 1, t + 2)$. The parameter t is the time in seconds since the motion started and the coordinates are in metres.

(a) Find the starting position of the object and its position after 2 seconds. [1]

(b) Eliminate t from the parametric representation and obtain the equation of the path in terms of x and y. What shape is this path? [3]

(c) Calculate the distance moved by the object along its path in the first 2 seconds. Give your answer to 2 decimal places. [2]

Question 4 – 8 marks

A ball is at rest on level ground. It is then kicked. The subsequent path of the ball until it first strikes the ground is described by the equation

$$y = \frac{x(40 - x)}{20},$$

where y is the height of the ball above the ground and x is the horizontal distance of the ball from the point on the ground from which it was kicked. Both x and y are in metres.

(a) When the ball hits the ground for the first time what is its distance from the point on the ground from which it was kicked? [1]

(b) Specify the range of values of x for which the above equation is valid. Hence define a function which gives the height of the ball above the ground. [2]

(c) The purpose of the kick is to put the ball over a bar which is 38 m from the point on the ground from which the ball is kicked. The bar is 3 m above the ground. Is the kick successful? Justify your answer. [2]

(d) What shape is the path of the ball. Use the properties of this curve to find the highest point above the ground reached by the ball. [3]

Question 5 – 5 marks

(a) Complete the table on the right below in order to trace the given algorithm. [4]

memory := ""
screen := "pan"
memory := *ADDLAST* (*LAST*(*screen*), *memory*)
screen := *DELLAST* (*screen*)
screen := *ADDLAST* ('i', *screen*)
screen := *ADDLAST* (*LAST*(*memory*), *screen*)
memory := *DELLAST* (*memory*)

memory	*screen*

(b) State the command which will change what is now shown on screen to "paint". [1]

Question 6 – 4 marks

The sum of a finite geometric sequence with first term a and common ratio a ($a \neq 1$) is given by the formula

$$\sum_{r=1}^{n} a^r = \frac{a - a^{n+1}}{1 - a}.$$

Describe what happens to this sum for large values of n, when

(a) $a = 0.9$; [2]

(b) $a = 1.1$. [2]

Question 7 – 5 marks

Figure 1 shows a Mathcad screen set up to show a logistic recurrence model for a given population.

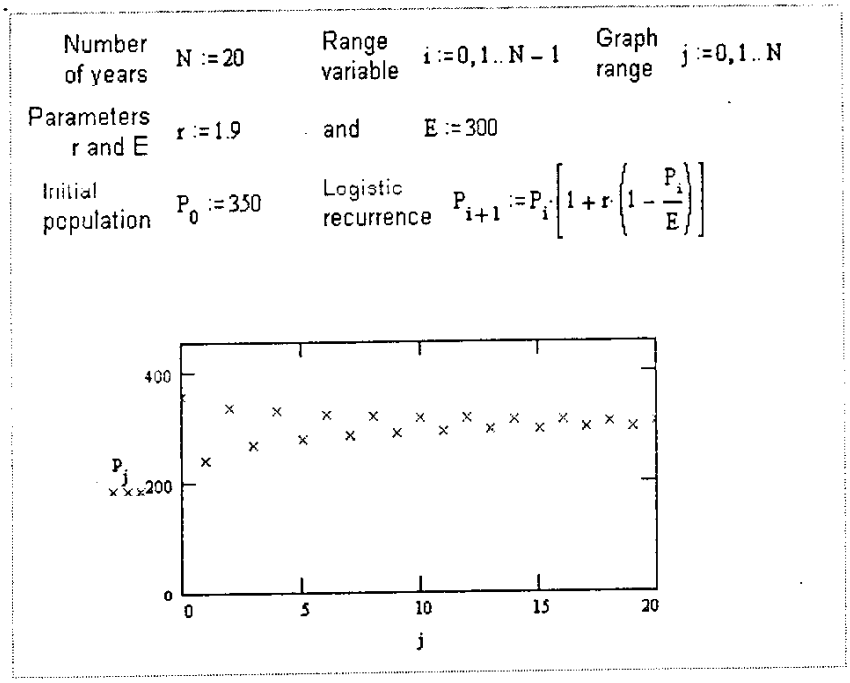


Figure 1

(a) Calculate the value of P_1 from the data given. [1]

(b) How would the Mathcad screen need to be adapted if the equilibrium population was 500 and the difference between the birth and death rates at low population levels was 1.5? [2]

(c) Describe the long-term behaviour of P_i indicating how this relates to the value of r. [2]

Question 8 – 6 marks

(a) For the matrices $\mathbf{P} = \begin{pmatrix} 1 & -5 \\ 3 & 0 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix}$

calculate each of the following

(i) $\mathbf{P} + \mathbf{Q}$ (ii) $\mathbf{QP}$. [2]

(b) The pair of equations

$$x_1 - 3x_2 = 9$$
$$2x_1 + 5x_2 = -4$$

is to be expressed in vector-matrix form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(i) Write down the matrix $\mathbf{A}$ and the vector $\mathbf{b}$. [1]

(ii) Write down the inverse of $\mathbf{A}$, and hence solve the equations. [3]

Question 9 – 5 marks

A jogger who runs along a straight street has position x metres at time t seconds, where

$$x = 100\left(1 - \cos\left(\frac{\pi t}{60}\right)\right) \quad (0 \leq t \leq 30).$$

(a) What distance does the jogger cover between $t = 0$ and $t = 30$? [1]

(b) What is the maximum velocity attained by the jogger during this time interval? [2]

(c) What is the maximum acceleration attained? [2]

Question 10 – 6 marks

(a) Differentiate the function

$$f(t) = 4t^5 - \sin(6t) \quad (t \in \mathbb{R}),$$

identifying any general rules which you use. [3]

(b) Find the indefinite integral of the function

$$g(x) = e^{-2x} + 3\sqrt{x} \quad (x > 0),$$

identifying any general rules which you use. [3]

Question 11 – 4 marks

The brakes are applied in a car which is travelling initially at $30\,\mathrm{m\,s^{-1}}$. The brakes provide a constant deceleration, and the car comes to a halt after 90 metres.

(a) Show that the magnitude of the deceleration is $5\,\mathrm{m\,s^{-2}}$. [2]

(b) What time elapses, from the moment of braking, before the car comes to rest? [2]

Question 12 – 5 marks

The temperature in a domestic room is 5°C at the moment when the house's heating system is first switched on. After a long time the room temperature settles at 25°C. The temperature θ°C at time t hours after the heating is switched on is to be modelled by a function of the form

$$\theta = A - B\exp(-0.575t).$$

(a) Sketch the graph of this temperature function, referring to the specific temperatures given above. [1]

(b) Find the appropriate values of the parameters A and B. [2]

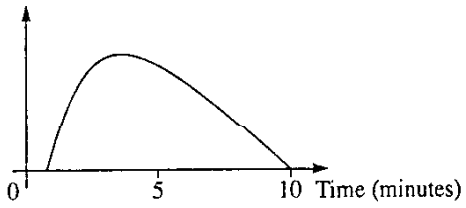
(c) According to this model, what is the time at which the temperature in the room reaches 10°C? [2]

Question 13 – 6 marks

(a) The curve drawn in the diagram below is to be used to model the variation in the time it takes to serve a customer in a certain bank. The total area under the curve is equal to 1.

Explain how the curve could be used to estimate the proportion of customers whose service time is less than two minutes. Use the diagram to illustrate your answer. (You do not need to draw another diagram.) [2]

Explanation



(b) The variation in the time taken to serve a customer may be modelled by a distribution with mean 3.5 minutes and standard deviation 1.8 minutes.

(i) Write down the mean and standard deviation of the sampling distribution of the mean service time for samples of 36 customers at this bank. [2]

(ii) Calculate a range of values within which the mean service time of approximately 95% of samples of 36 customers will lie. [2]

Question 14 – 4 marks

The labels on bottles of olive oil produced by a company say that the bottles contain 1 litre. The mean contents of a sample of 50 bottles taken from a day's production is $1002\,\text{cm}^3$ and the sample standard deviation is $10\,\text{cm}^3$.

(a) Calculate a 95% confidence interval for the mean contents of all bottles from the day's production. [2]

(b) Hence explain whether or not it is likely that the mean contents of all the bottles from the day's production is less than 1 litre. [2]

Question 15 – 6 marks

The number of eggs in a nest (the clutch size) was recorded for 286 pairs of lapwings; 209 pairs were breeding on agricultural grassland and the other 77 pairs were breeding on land cultivated annually. (A lapwing is a bird which nests at ground level.) The sample mean and sample standard deviation of the clutch size were calculated separately for the lapwings nesting on the two types of agricultural land. The statistics are given in the table below.

	Grassland	Cultivated land
Sample size	209	77
Mean	3.80	3.63
Standard deviation	0.72	0.53

A two-sample z-test is to be used to investigate whether, in general, there is a difference between the mean clutch size of lapwings breeding on grassland and the mean clutch size of lapwings breeding on cultivated land. The hypotheses are as follows:

$$H_0 : \mu_G = \mu_C,$$
$$H_1 : \mu_G \neq \mu_C,$$

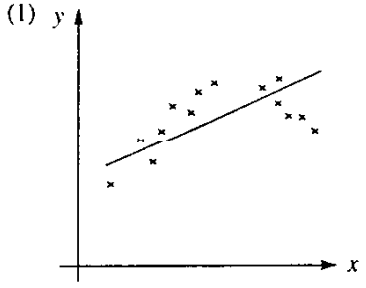
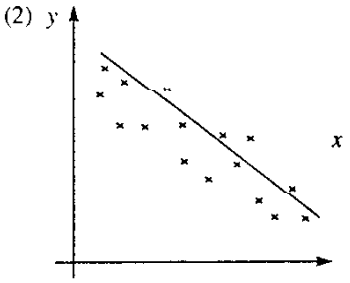
where μ_G is the mean clutch size of lapwings breeding on grassland and μ_C is the mean clutch size of lapwings breeding on cultivated land.

(a) Calculate the z-test statistic. [2]

(b) What conclusions can you draw from the results of this hypothesis test about the clutch sizes of lapwings breeding on agricultural grassland and of lapwings breeding on cultivated land? [4]

Question 16 – 4 marks

(a) For each of the diagrams below, give one reason why the line shown does NOT fit the data well. [2]

Diagram	Reason
(1) 	(1)
(2) 	(2)

(b) For each plot above, draw on the diagram either a line or a curve which you think would be a better model for the relationship between the variables x and y. *No calculations are required: you should fit your lines or curves by eye.* [2]

PART II

Instructions

(i) *You should attempt not more than* **TWO** *questions from this part of the examination.*

(ii) *Each question in this part carries 10% of the marks.*

(iii) *You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.*

(iv) *Show all your working.*

Question 17

Ms Holly decides to buy a new fitted kitchen. The price of the kitchen she chooses is £6000, including fitting and all associated work. The vendor has an agreement with a bank to offer personal loans with a monthly rate of interest of 1.5%. Ms Holly decides to take out such a loan and agrees to make a monthly repayment of £150. There is no initial deposit.

The first repayment on the loan is to be made one calendar month after the completion of the installation and further payments become due at monthly intervals thereafter. Repayments are made by direct debit, so Ms Holly's bank account is debited by £150 on each scheduled payment due date. Interest is calculated at the close of business on the day before the monthly payment is taken. The interest is 1.5% of the balance in the account after the previous payment was made (or, in the case of the first month, 1.5% of the amount of the loan).

If £b_0 is the cost of the kitchen and £b_n is the balance of the account after the nth montly payment has been made.

(a) Give the recurrence system for b_n. [1]

(b) Find the closed form for b_n, and hence calculate the amount still owing after 36 payments have been made. [3]

(c) Write down a formal definition of a function f, which gives the balance, b_n (in £) in terms of n, the number of payments made. [2]

(d) Find the inverse function of f, and hence calculate the number of payments required to pay off the loan, assuming that the monthly repayment stays fixed at £150 (except possibly the last payment). It is not necessary to calculate the exact amount of the last payment. [4]

(You may find the following information helpful in answering this question:
The closed form for a general recurrence relation

$$u_0 = a, \quad u_{i+1} = ku_i + c.$$

is $u_n = \left(a + \frac{c}{k-1}\right) \times k^n - \frac{c}{k-1}$ with $n = 0, 1, 2, \ldots$)

Question 18 – 10 marks

A population of seabirds may be divided up into juveniles (up to one year old), immatures (between 1 and 3 years old), and adults (over 3 years old). A model of the population is constructed and expressed in the form $\mathbf{P}_{n+1} = \mathbf{M}\mathbf{P}_n$, where

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 0.9 \\ 0.7 & 0.5 & 0 \\ 0 & 0.4 & 0.8 \end{pmatrix} \text{ and } \mathbf{P}_n = \begin{pmatrix} J_n \\ I_n \\ A_n \end{pmatrix}$$

and J_n, I_n and A_n are the numbers of juveniles, immatures and adults at the end of year n.

(a) Following this model express

(i) J_{n+1} (ii) I_{n+1} (iii) A_{n+1}

in terms of the previous values J_n, I_n and A_n. [3]

(b) State what assumptions have been made about

(i) the birth rate for the subpopulation of adults;

(ii) the death rate for the subpopulation of adults;

(iii) the death rate for the subpopulation of immatures. [3]

(c) According to Mathcad,

$$\mathbf{M}^{-1} = \begin{pmatrix} 1.6 & 1.44 & -1.8 \\ -2.24 & 0 & 2.52 \\ 1.12 & 0 & 0 \end{pmatrix}.$$

At the end of 1998 the estimated numbers of juveniles, immatures and adults were, respectively: 500, 900 and 850. Calculate the estimated numbers of birds in each category

(i) at the end of 1999;

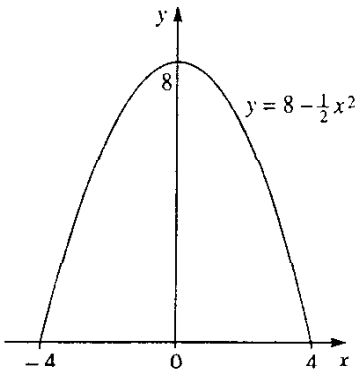
(ii) at the end of 1997. [4]

Question 19

The figure below shows the uniform cross-section of a futuristic building design, whose outer surface is given by the equation

$$y = 8 - \tfrac{1}{2}x^2 \quad (-4 \le x \le 4),$$

with all lengths being measured in metres.

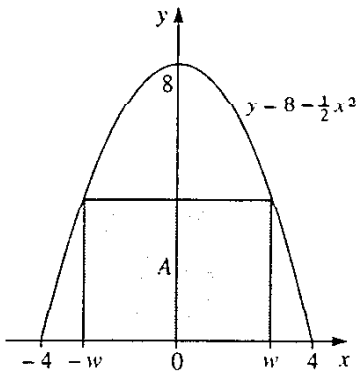


(a) (i) Find the area of the cross-section shown, within the outer surface. [3]

(ii) Check your answer for this area by comparing it with the areas of

- the smallest rectangle which encloses the cross-section, and
- the largest triangle which can be inscribed within the cross-section. [1]

(b) The architect intends the living quarters within the building to have a uniform rectangular cross section, of area A m^2 and width $2w$ metres, as shown in the figure below.



(i) Express A in terms of w. [1]

(ii) Find the value of w for which A is a maximum, and the corresponding values of A and of the height of the living quarters. [4]

(iii) Express the maximum cross-sectional area of the living quarters as a proportion of the overall building area found in part (a)(i). [1]

Question 20

(a) Two people each pick a number between 1 and 10 (inclusive). Each person chooses a number at random, that is, so that no number is more likely to be chosen than any other number.

(i) Find the probability that they both choose the number 6.

(ii) This process is repeated ten times. Find the probability that they do not both choose the number 6 on any of the ten occasions. What is the probability that they both choose the number 6 on at least one of the ten occasions? [4]

(b) Tom, Dick and Harriet each pick a number between 1 and 10 (inclusive) at random.

(i) What is the probability that Tom and Dick choose different numbers?

(ii) Find the probability that Tom, Dick and Harriet all choose different numbers.

(iii) Find the probability that at least two of them choose the same number.

(iv) Find the probability that they all choose the same number.

(v) Find the probability that exactly two of them choose the same number. [6]

[END OF QUESTION PAPER]