MST121/D

The Open University

First Level Course Examination 1997

Using Mathematics

Tuesday, October 14, 1997 2.30 pm – 5.30 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I you should attempt as many questions as you can, writing your answers *in the spaces provided* inside this examination paper. You should attempt not more than **TWO** questions in Part II. Your answers to this part should be written in the answer book provided.

80% of the available marks are assigned to Part I and 20% to Part II. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part I before starting Part II.

Graph paper is available from the invigilator, if you feel it would assist you in answering questions.

At the end of the examination

Check that you have completed the grid below, and have written your name, personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

Examination No.								
Surname (BLOCK LETTERS)					Initials			
Personal Identifier								

This question paper must NOT be removed from the examination room. It must be handed to the invigilator at the end of the examination.

The University reserves the right not to mark your script if you fail to follow these instructions.

Attach this examination paper to the FRONT of the answer book(s) in which you have answered questions from Part II.

NB: To aid downloading I have eliminated many of the spaces provided for writing answers.

PART I

Instructions

(i) *You should attempt as many questions as you can in this part of the examination.*

(ii) *Part I carries 80% of the available examination marks. Each question indicates how many of these marks are allocated to it.*

(iii) *You should, as far as possible, record your answers to each question in this part in the space provided on the question paper. You are strongly advised to show all your working, including any rough working. If you need extra space then you may continue your working in a separate answer book. If you do this, make sure that your work is clearly labelled.*

Question 1 – 3 marks

The recurrence system

$$x_1 = 3,\ x_{n+1} = x_n + 0.9 \quad (n = 1, 2, 3, \ldots)$$

generates a sequence.

(a) Write down the first 4 terms of the sequence. [1]

(b) State what kind of sequence this is and give its closed form, [2]

Question 2 – 3 marks

The formula

$$x_{n+1} = 3(0.9)^n \quad (n = 0, 1, 2, \ldots)$$

generates a sequence.

(a) Write down the first 4 terms of the sequence. [1]

(b) State what kind of sequence this is and express it as a recurrence system. [2]

Question 3 – 8 marks

The demand (D, in thousands) for and supply (S, in thousands) of an item are modelled by functions expressed in terms of the price P, where P is in units of £1000. These functions are $D = 10 - P$, $S = 2P^2 - 12$, for suitable values of P.

(a) Give the values of D and S when $P = 4$. [2]

(b) Give the range of possible values for P, and reasons for your answer. [3]

(c) Find the values of P for which $D = S$ and hence state to the nearest £, the price of the item for which supply and demand are equal. [3]

Question 4 – 6 marks

A model for the proportion of vocabulary in common for two languages which are diverging is given by the function

$$\begin{aligned} f : \mathbb{R}^+ &\longrightarrow \mathbb{R} \\ x &\longmapsto e^{-0.03x} \end{aligned}$$

where x is the time in centuries since separation of the corresponding populations, and $f(x)$ is the proportion of vocabulary in common.

(a) Use this model to find the proportion of vocabulary in common for two languages which have diverged for

(i) 200 years;

(ii) 2000 years. [2]

(b) Give the rule for the inverse function f^{-1}. [2]

(c) If two languages have 60% of their vocabularies in common, for how long, according to the model, have they been diverging? [2]

Question 5 – 4 marks

(a) Express in words the condition

$$AND\ (AND(-10 \leq x, x \leq 3), NOT(x = 0)).$$ [2]

(b) Express in Boolean notation the condition

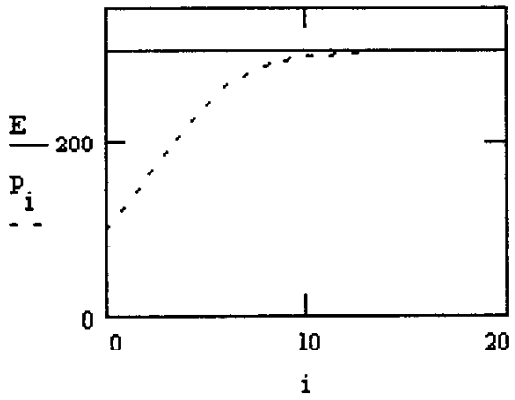
x is positive but not equal to 3. [2]

Question 6 – 5 marks

The size of a population of geese is modelled by the sequence P_i generated by the logistic recurrence relation

$$P_{i+1} - P_i = rP_i\left(1 - \frac{P_i}{E}\right) \quad (i = 0, 1, 2, \ldots)$$

for specific values of E, r and P_0. This is graphed below.



(a) By inspecting the graph write down approximate values for E and P_0. [2]

(b) Using these values and given that $r = 0.4$, write down the specific recurrence system which models the flock. [1]

(c) Find the corresponding value of P_1. [2]

Question 7 – 5 marks

Each of the formulas below generates a sequence. For each sequence, say whether or not it converges, and if it does, state to what limit. Explain the reasoning behind your answers, very briefly.

(a) $u_i = 2 + (-1)^i$ with $i = 1, 2, 3, \ldots$ [1]

(b) $v_i = 2 + (-0.5)^i$ with $i = 1, 2, 3, \ldots$ [2]

(c) $w_i = 2 - \frac{1}{i^2}$ with $i = 1, 2, 3, \ldots$ [2]

Question 8 – 6 marks

(a) For the matrices $\mathbf{C}$ and $\mathbf{D}$ below, calculate $\mathbf{C} + \mathbf{D}$ and $\mathbf{DC}$.

$$\mathbf{C} = \begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 5 & 1 \\ -4 & 2 \end{pmatrix}$$ [2]

(b) The pair of equations

$$4x_1 - x_2 = 11$$
$$2x_1 + 3x_2 = -5$$

is to be expressed in the vector-matrix form $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(i) Write down the matrix $\mathbf{A}$ and the vector $\mathbf{b}$. [1]

(ii) Write down the inverse of $\mathbf{A}$, and hence solve the equations. [3]

Question 9 – 6 marks

A manufacturer plans to produce a closed box of volume 500 cm^3. It is to be made in the shape of a cuboid with square base and height h cm. It is noted that its external surface area s cm^2 is given by the formula

$$s = 2x^2 + \frac{2000}{x} \quad (x > 0)$$

where x cm is the length of the side of the square base. It is intended to reduce the cost of materials by minimising the surface area, provided this gives a box of suitable dimensions.

(a) Write down an expression for the volume of the box in terms of h and x. Hence find an expression for h in terms of x. [1]

(b) Find the value of x for which s is a minimum. [3]

(c) Evaluate this minimum value of s. [1]

(d) Find the value of h which corresponds to this minimum value of s. [1]

Question 10 – 6 marks

(a) Differentiate the function

$$f(x) = 2x^4 + 5\exp(3x) \quad (x \in \mathbb{R}),$$

identifying any general results which you use. [3]

(b) Find the indefinite integral of the function

$$g(t) = 3\cos(4t) - \frac{5}{t} \quad (t > 0),$$

identifying any general results which you use. [3]

Question 11 – 4 marks

(a) Find the x-coordinates of the points at which the graph of the function $f(x) = -x^2 + 4x - 3$ crosses the x-axis. [2]

(b) Find the area bounded by the graph of the function $f(x) = -x^2 + 4x - 3$ and the x-axis. [2]

Question 12 – 4 marks

The temperature within the production area of a factory varies during the working day as a result of both solar heating and the intensity of activity inside the factory. The factory is in operation between 6.00 am and 8.00 pm. On a typical summer day the temperature is lowest at 6.00 am, when it is 17 °C, reaches 22 °C at 10.00 am and is at its highest at 2.00 pm, when it is 27 °C.

An equation of the form

$$H = A + B\sin(kt)$$

is to be used to model the variation on summer days of the temperature H°C with time t hours ***since 10.00 am***. In this equation A, B and k are constants.

(a) Sketch the graph of this equation and mark on it the given temperature and time values. [2]

(b) Write down appropriate values of A and B. [1]

(c) Find an appropriate value of k. [1]

Question 13 – 7 marks

Two fair six-sided dice are rolled together repeatedly.

(a) Write down the probability of obtaining a double five in a single roll of the two dice; that is, write down the probability that the two dice both land with the face labelled five uppermost. [2]

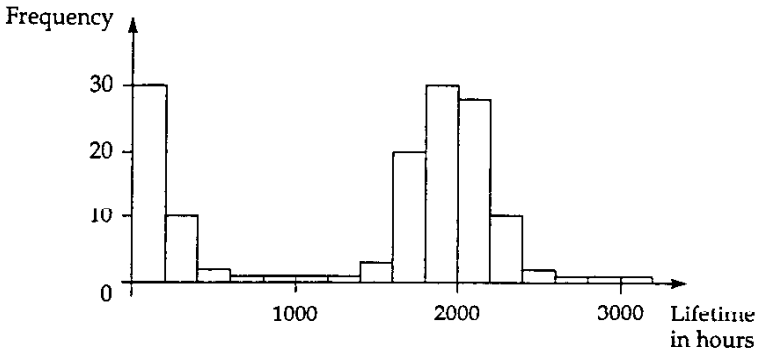
(b) Find the probability that none of the first ten rolls of the two dice results in a double five. [2]

(c) Find the probability that there is at least one double five in the first ten rolls of the two dice. [1]

(d) The two dice are rolled until a double five is obtained. How many times would you expect the two dice to be rolled – that is, what is the mean number of rolls of two dice required to obtain a double five? [2]

Question 14 – 4 marks

The frequency diagram below represents the lifetimes of a sample of light bulbs produced by a company.



(a) Give one reason why it would not be appropriate to use a normal distribution to model the variation in the lifetimes of light bulbs produced by the company. [1]

(b) Sketch a curve which you think might provide a reasonable model for the variation in the lifetimes of light bulbs produced by the company. You should include a horizontal scale on your sketch, but there is no need to include a vertical scale. [1]

(c) With the assistance of a sketch, explain how you might use your model to estimate the proportion of all the light bulbs produced by the company that last for less than 500 hours. [2]

Question 15 – 7 marks

(a) Records show that for a particular brand of light bulbs produced by a company the lifetimes have mean 1600 hours and standard deviation 125 hours.

(i) Write down the mean and standard deviation of the sampling distribution of the mean for samples of size 100 from this population. [2]

(ii) Calculate a range of values within which the mean lifetime (in hours) of approximately 95% of samples of 100 light bulbs will lie. [2]

(b) A new method of production is introduced. To check the quality of light bulbs produced by this method, the lifetimes of a sample of 100 bulbs are obtained. The sample mean is 1580 hours and the sample standard deviation is 80 hours.

(i) Calculate an approximate 95% confidence interval for the mean lifetime (in hours) of bulbs produced by this method. [2]

(ii) What can the manufacturer conclude from this confidence interval about the mean lifetime of light bulbs produced by this new method as compared with the old method? [1]

Question 16 – 2 marks

In a study of physical measurements within families, data are collected for a large number of families on the heights in centimetres of fathers and sons. Data on the heights of the fathers (x) and the heights of their sons (y) are used to calculate a least squares fit line to model the relationship between the heights of fathers and their sons. The equation of this line is

$$y = 0.52x + 86.$$

According to this model, are the sons of 170 cm tall men taller or shorter, on average, than their fathers, and by how much? [2]

PART II

Instructions

(i) *You should attempt not more than* **TWO** *questions from this part of the examination.*

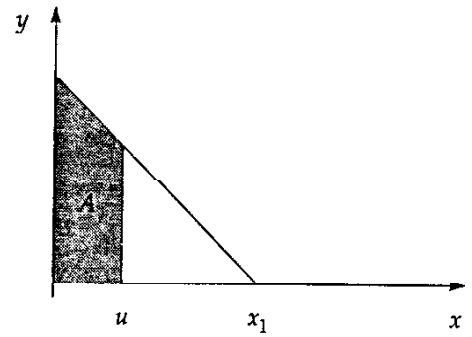
(ii) *Each question in this part carries 10% of the marks.*

(iii) *You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.*

(iv) *Show all your working.*

Question 17

In the figure below; the line with parametric coordinates $(t, 20 - t)$ intersects the x-axis at $x = x_1$. The shaded area A is bounded by this line, by the two axes and by the line $x = u$.



(a) Eliminate the parameter t to give an equation for the line of the form

$y =$ expression in x. [1]

(b) Find x_1. [1]

(c) Find an expression in terms of u for the shaded area A. Treating u now as variable, express A as a function with domain $[0, x_1]$. [4]

(d) What is A_1, the value of A when $u = x_1$? [1]

(e) For what value of u is A equal to half of A_1? [3]

Question 18

(a) Draw an expression tree for each of the following Mathcad expressions.

(i) $\dfrac{y^2 + y + 4}{2 \cdot y}$ [2]

(ii) $\dfrac{y + 1}{2} + \dfrac{2}{y}$ [2]

(b) (i) Trace the algorithm below using a suitable table.

memone := *y*
memtwo := *memone* × *memone*
memthree := *memone* + *memtwo*
memthree := *memthree* + 4
memone := 2 × *memone*
memthree := *memthree* ÷ *memone*
The calculated value is in *memthree*. [4]

(ii) State whether this algorithm will evaluate the expressions above, giving brief reasons for your answer in each case. [2]

Question 19

In a fairground game, the velocity $v(t)\,\text{ms}^{-1}$, at time t seconds, of a small ball moving along a straight track while attached to a spring is modelled by the function

$$v(t) = \frac{1}{3}\sin(3t).$$

(a) Find the acceleration $a(t)\,\text{ms}^{-2}$ of the ball. What is the acceleration when $t = 0$? [2]

(b) Find the position $s(t)$ metres of the ball, given that $s = \frac{1}{3}$ m at time $t = 0$. [3]

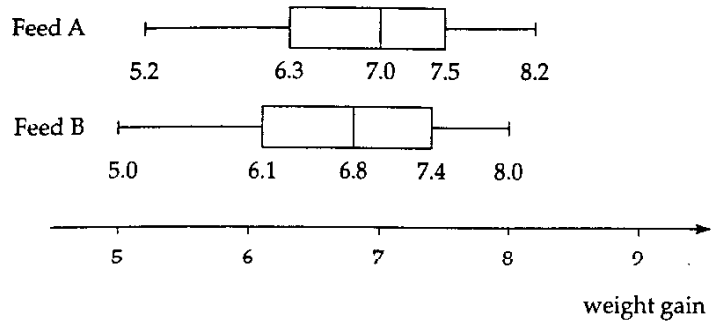
(c) Show that the acceleration of the ball can be expressed in terms of its position by the equation $a = -9s + 4$. [3]

(d) Describe in words the motion of the ball for $t > 0$. [2]

Question 20

In order to compare two types of feed for young turkeys (feed A and feed B), two samples of turkeys were selected from a flock. For a given period, one sample of turkeys was given only feed A and the other was given only feed B. At the end of the period, the weight gain in kilograms of each turkey was recorded.

(a) Boxplots for the weight gains in kilograms of the two samples of turkeys are shown below.



Use the boxplots to compare the weight gains of the turkeys given feed A and the turkeys given feed B. Your comparison should include comments on the relative size of the weight gains of the turkeys given the two feeds and on the variation in the weight gains of the turkeys given the two feeds. [3]

(b) The sample mean and sample standard deviation of the weight gains in kilograms for each of the two samples of turkeys are given in the table below.

	Feed A	Feed B
Sample size	72	80
Mean	7.0	6.7
Standard deviation	0.88	0.96

It is proposed to use a two-sample z-test to investigate whether, in general, there is a difference between the mean weight gain of turkeys when given only feed A and the mean weight gain of turkeys when given only feed B.

(i) Use symbols to write down concisely appropriate null and alternative hypotheses for the test. Explain the meanings of the symbols that you use.

(ii) Calculate the test statistic.

(iii) What conclusions can you draw from the results of this hypothesis test about the weight gains of turkeys given only feed A and of turkeys given only feed B? [7]

[END OF QUESTION PAPER]