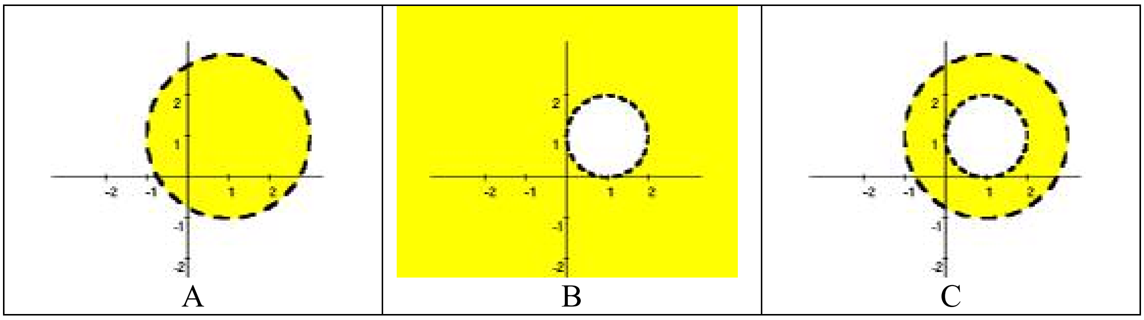
[[I should not include most of the unit references in the answers in an exam. They are there to help us find the relevant information in the handbook or units. If anyone would like to give me a copy of the handbook then I should be grateful.]]

Question 1

(a)



(b) A is an open ball which is an open set (Unit A2 Theorem 4.1).
$B_{d^{(2)}}[(1,1),1]$ is a closed set (Unit A4, Theorem 1.2). As B is the complement of a closed set then it is open (Unit A4, Th. 1.1).
C is the intersection of two open sets so it is also an open set (Unit A2 Theorem 4.6).

(c) Since C is an open set then Int(C) = C (Unit A4 Corollary 4.3).

Let $x \in S = S_{d^{(2)}}((1,1),1) \cup S_{d^{(2)}}((1,1),2)$.
As every neighbourhood of x contains points of C then each $x \in S$ is a closure point of C.

As $C \cup S = B_{d^{(2)}}[(1,1),2] \cap B_{d^{(2)}}((1,1),1)^C$ is the intersection of closed sets then it is closed (Unit A4 Theorem 1.4).
As $C \cup S$ only contains closure points of C and is closed then
$Cl(C) = C \cup S$ (Unit A4 Theorem 2.5).

By Unit A4 Corollary 4.4, $Ext(C) = Cl(C)^C$.
Therefore $Ext(C) = B_{d^{(2)}}[(1,1),2]^C \cup B_{d^{(2)}}((1,1),1)$.

By Unit A4 Corollary 4.6, $Bd(C) = (Int(C) \cup Ext(C))^C$

Hence $Bd(C) = S_{d^{(2)}}((1,1),1) \cup S_{d^{(2)}}((1,1),2)$.

Question 2

(a) The function $f : X \rightarrow \mathbb{Z}$ is continuous if $f^{-1}(U) \in T_X$ whenever $U \in T_{\mathbb{Z}}$.
(Unit A4, Section 1.3, Definition)

Let $U \in \mathcal{T}_{\mathbb{Z}}$.

If $0 \in U$ then $1 \in U$. So $f^{-1}(U) = X \in \mathcal{T}_X$.
If $0 \notin U$ and $1 \in U$ then $f^{-1}(U) = \{a, b\} \in \mathcal{T}_X$.
If $0 \notin U$ and $1 \notin U$ then $f^{-1}(U) = \phi \in \mathcal{T}_X$.

Since $f^{-1}(U) \in \mathcal{T}_X$ for all $U \in \mathcal{T}_{\mathbb{Z}}$ then f is $(\mathcal{T}_X, \mathcal{T}_{\mathbb{Z}})$-continuous.

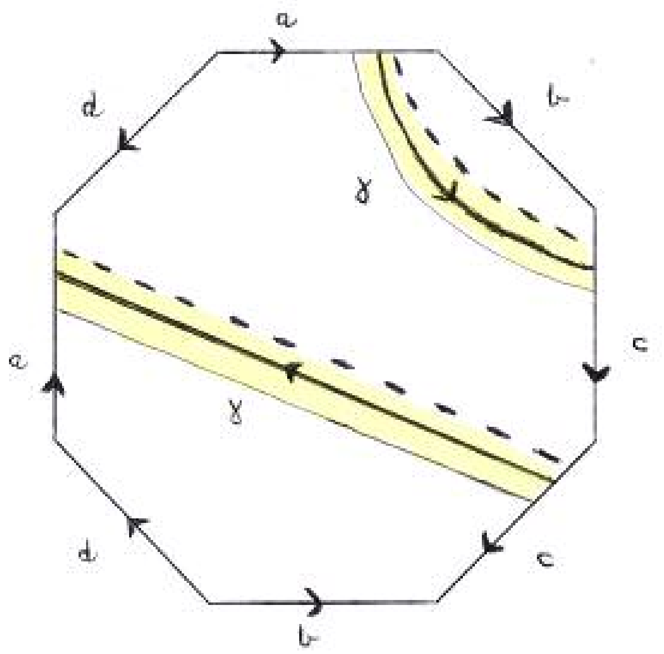
(b) $\{1\} \in \mathcal{T}_{\mathbb{Z}}$.

$g^{-1}(\{1\}) = \{b, c\} \notin \mathcal{T}_X$.

Since it is not true that $g^{-1}(U) \in \mathcal{T}_X$ whenever $U \in \mathcal{T}_{\mathbb{Z}}$ then
g is not $(\mathcal{T}_X, \mathcal{T}_{\mathbb{Z}})$-continuous.

Question 3

(a)



The thickened neighbourhood is a cylinder as it has two boundaries. [[Unit B2, Section 2.3]]

(b) The surface is not orientable.

There is a pair of identified edges (a or c) that occur twice in the same sense (Unit B2, Theorem 2.1).

(c) $abccb^{-1}dad^{-1} = 1$. [[Unit B2, Section 3.3]]

Question 4

(a) For an orientable surface (Unit B3 Section 3.1)

$$\chi = 2 - 2(\text{number of handles}) - (\text{number of holes}).$$

Since the number of holes is zero then χ must be even. Therefore there is no orientable surface with $\chi = -5$.

OR

Unit B3, Th. 4.9 says there is an orientable surface with Euler characteristic $\chi \leq 2$ if and only if $\chi + \beta$ is even, where β is the boundary number. Therefore there cannot be a surface with $\chi = -5$ without a boundary ($\beta = 0$) as $\chi + \beta$ is not even.

(b)

Unit B3 Theorem 4.10 says that there is a non-orientable surface with $\chi = -5$ and boundary number $\beta \geq 0$ if and only if

$$-5 + \beta \leq 1$$

and the surface is $(2 - (-5 + \beta)\,\mathbb{P} \# \beta\,\mathbb{D}$.

This gives the following possible non-orientable surfaces with $\chi = -5$.

β	Surface
0	$7\,\mathbb{P}$
1	$6\,\mathbb{P} \# 1\,\mathbb{D}$
2	$5\,\mathbb{P} \# 2\,\mathbb{D}$
3	$4\,\mathbb{P} \# 3\,\mathbb{D}$
4	$3\,\mathbb{P} \# 4\,\mathbb{D}$
5	$2\,\mathbb{P} \# 5\,\mathbb{D}$
6	$\mathbb{P} \# 6\,\mathbb{D}$

Question 5

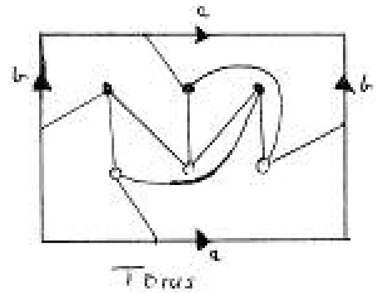
(a) Using Ringel's theorem for the orientable genus (Unit B4, Theorem 3.7) gives

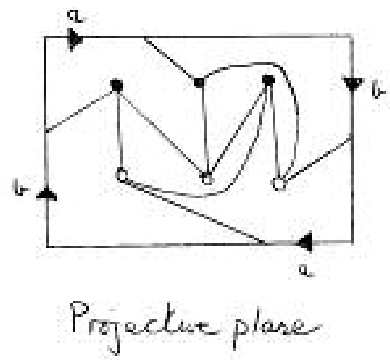
$$g(K_{3,3}) = \lceil \tfrac{1}{4}(3 - 2)(3 - 2) \rceil = \lceil \tfrac{1}{4} \rceil = 1.$$

Using Ringel's theorem for the non-orientable genus (Unit B4, Theorem 3.11) gives

$$q(K_{3,3}) = \lceil \tfrac{1}{2}(3 - 2)(3 - 2) \rceil = \lceil \tfrac{1}{2} \rceil = 1.$$

(b)





Question 6

(a) (i) Let $\mathcal{S}$ be an arbitrary open cover of (0, 1]. [[C2, Sect. 1.1]]

Since $1 \in (0, 1]$ then the covering $\mathcal{S}$ contains an open set which contains 1. The only open sets which contain 1 also have (0, 1] as a subset. Therefore any covering of (0, 1] contains a one-set subcover.

Hence (0, 1] is compact for $\mathcal{T}$.

(a) (ii) Let $\mathcal{S} = \{[0, n) : n \in \mathbb{N}\}$.

If $x \in [0, \infty)$ then $x \in [0, n)$ for some $n \in \mathbb{N}$. Therefore $\mathcal{S}$ is an open cover for X.

Assume that there is a finite subset of $\mathcal{S}$ which is a cover for X. Then we can write this covering as $\mathcal{B} = \{[0, n_1), [0, n_2), \ldots, [0, n_k)\}$ for some $k \in \mathbb{N}$.

If $x \geq \max \{n_i : 1 \leq i \leq k\}$ then $x \notin \mathcal{B}$. Therefore there is no finite subset of $\mathcal{S}$ which covers X.

Hence X is not compact for $\mathcal{T}$. [[C2, Sect. 2.1]]

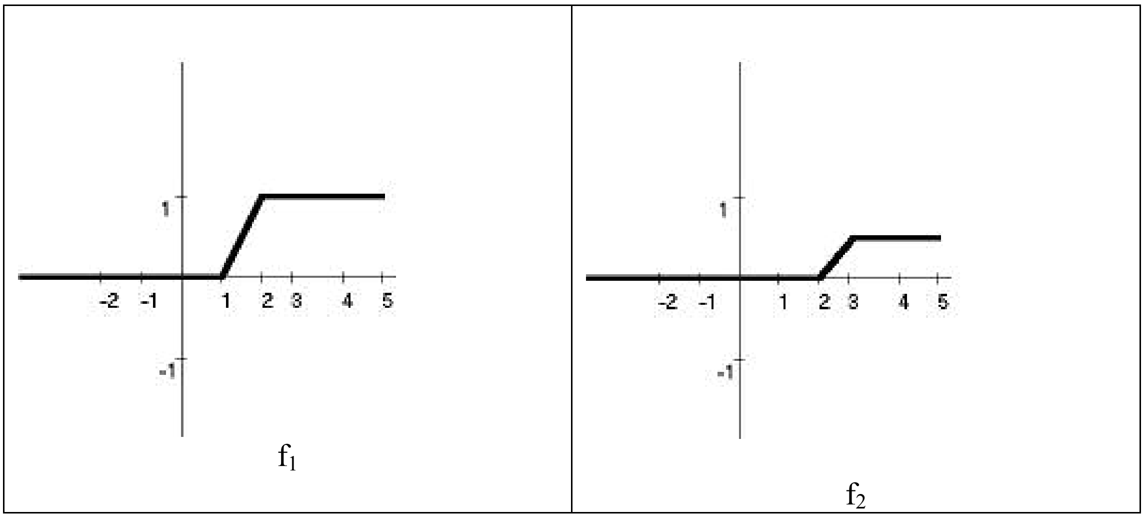
(b) If $x > 0$ then any open set which contains x also contains the point 0.

Since there are no disjoint open sets which contain both 0 and x then $(X, \mathcal{T})$ is not Hausdorff.

[[Alternatively. As {0} is not a closed set then X is not Hausdorff (Unit C2 Theorem 3.2).]]

Question 7

(a)



f_1 f_2

(b) For any $x \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that $x < N$.

For all $n \geq N$, $f_n(x) = 0$.
Therefore $f_n(x) \to 0$ as $n \to \infty$ for $x \in \mathbb{R}$.

So the pointwise limit function of the sequence (f_n) is the zero function
$f : \mathbb{R} \to \mathbb{R}$
$f(x) = 0$ for all $x \in \mathbb{R}$.

(c) Let f be function defined above.

I shall check the conditions in the definition of Uniform Convergence given in Unit C3 Section 3.

Condition (a).
For $n \in \mathbb{N}$ we have $0 \leq | f_n(x) - f(x) | \leq 1/n$.
Therefore $f_n - f$ is bounded for all $n \in \mathbb{N}$.

Condition (b).
Let $M_n = \sup \{ | f_n(x) - f(x) | : x \in \mathbb{R} \}$.

The sequence $(M_n) = (1/n)$ is a null sequence. [[A1, Sect. 1.3]]

Since conditions (a) and (b) are satisfied then f is the uniform limit of (f_n).

Question 8

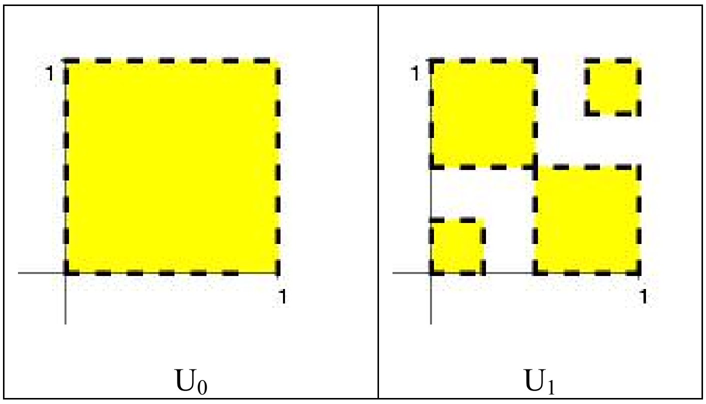
(a) Suitable similarities $S_1, S_2, S_3, S_4 : \mathbb{R}^2 \to \mathbb{R}^2$ are

$S_1(x, y) = (x/4, y/4)$
$S_2(x, y) = (x/2, (1 + y)/2)$
$S_3(x, y) = ((1 + x)/2, y/2)$
$S_4(x, y) = ((3 + x)/4, (3 + y)/4)$.

S_1 and S_4 have similarity ratio 1/4. [[C5, Sect. 1.1]]
S_2 and S_3 have similarity ratio 1/2.

(b)

Let U_0 be the interior of K_0. U_0 is a non-empty open bounded set. The following diagrams show that S_1, S_2, S_3, S_4 satisfy the open set condition as the sets $S_1(U_0)$, $S_2(U_0)$, $S_3(U_0)$, $S_4(U_0)$ are disjoint and their union is a subset of U_0.



(c) Let K_S be the invariant set of the similarities.
Since the similarities satisfy the open set condition then, using Unit C5 Theorem 3.1,

$$\dim K_S = s$$

where s is the solution of $2\left(\frac{1}{4}\right)^s + 2\left(\frac{1}{2}\right)^s = 1$.

Putting $y = \left(\frac{1}{2}\right)^s$ we get the equation $2y^2 + 2y - 1 = 0$.

As $y > 0$ this has the solution $y = \frac{-2+\sqrt{4+8}}{4} = \frac{\sqrt{3}-1}{2}$.

So $\dim K_S = s = -\frac{\log y}{\log 2} = \frac{\log 2 - \log\left(\sqrt{3}-1\right)}{\log 2} = 1 - \frac{\log\left(\sqrt{3}-1\right)}{\log 2}$. [[About 1.45]]

Question 9

(a) The method of inserting vertices (Unit B2, Section 3.2) to the edge expression gives

$$\begin{array}{cccccccccccccccccc} P & a & P & b & P & a^{-1} & P & c & P & d & P & c^{-1} & P & e & P & f & P & e^{-1} & P & b^{-1} & P & d^{-1} & P & f^{-1} & P \\ 1 & & 5 & & 4 & & 1 & & 8 & & 7 & & 2 & & 11 & & 10 & & 3 & & 6 & & 9 & & 1 \end{array}$$

The number below the vertices show the order in which they were added.

1 - a starts at P;	2 - c starts at P;	3 - e starts at P;
4 - b ends at P;	5 - a ends at P;	6 - b starts at P;
7 - d ends at P;	8 - c ends at P;	9 - d starts at P;
10 - f ends at P;	11 - e ends at P.	

There is one vertex, one face and 6 edges. So $\chi = 1 - 6 + 1 = -4$.

There are no edges which only occur once in the edge equation so the boundary number $\beta = 0$.

There are no symbols which occur twice in the same direction so the surface is orientable (Unit B2, Th. 2.1). Hence $\omega = 0$.

(b)

I shall use the strategy for obtaining the canonical form given in Unit B3 Section 3.

Step 1. Assemble cross-caps.

There are no symbols repeated in the same sense so there are no cross-caps.

Step 2. Assemble handles.

Using the Assembling Lemma we find

$$\underline{a\,b\,a^{-1}}\,c\,d\,c^{-1}\,e\,f\,e^{-1}\,\underline{b^{-1}}\,d^{-1}\,f^{-1} = 1$$

$$\to \quad a\,b\,a^{-1}\,b^{-1}\,\underline{c\,d\,c^{-1}}\,e\,f\,e^{-1}\,\underline{d^{-1}}\,f^{-1} = 1$$

$$\to \quad a\,b\,a^{-1}\,b^{-1}\,c\,d\,c^{-1}\,d^{-1}\,e\,f\,e^{-1}\,f^{-1} = 1.$$

Step 3. Assemble holes.

There are no unrepeated letters so there are no holes.

Therefore the canonical form is $a\,b\,a^{-1}\,b^{-1}\,c\,d\,c^{-1}\,d^{-1}\,e\,f\,e^{-1}\,f^{-1} = 1$.

There are no cross-caps so the surface is orientable and $\omega = 0$.

The orientable surface has

$$\chi = 2 - 2(\text{number of handles}) - (\text{number of holes})$$
$$= 2 - 2 * 3 - 0 = -4.$$

There are no holes so $\beta = 0$.

These values agree with those found in part (a).

(c) $3\mathbb{T}$ (Unit B3, Th. 4.2)

(d) The canonical form is

$$a\,b\,a^{-1}\,b^{-1}\,c\,d\,c^{-1}\,d^{-1}\,e\,f\,e^{-1}\,f^{-1}\,g\,h^{-1}\,g = 1.$$

When applying the procedure to obtain the canonical form

Step 1. Unchanged

Step 2, $g\,h^{-1}\,g$ will occur at the end of block F and will remain there when the Assembling Lemma is used.

Step 3. As the hole is already in the right format no action is required.

Question 10

[[I have problems with convincing myself that my answers to this type of question are correct. Please give me reassurance or show me the error of my ways.]]

(a)(i) The number of faces, F = 2.

There are 5F = 10 edges round the faces, but these have been counted twice. Hence the number of distinct edges, E = 5. [[Unit B2, Th. 1.1 Proof]]

(a)(ii) There are Vj edges meeting at the vertices, but each edge ends at 2 vertices. Therefore Vj = 2E = 10.

(a)(iii) $\chi = V - E + F = \frac{10}{j} - 5 + 2 = \frac{10}{j} - 3$.

A regular subdivision must have $j \geq 2$ (Unit B2 Th. 1.1).
The table below shows the possible subdivisions allowed.

j	2	5	10
V	5	2	1
χ	2	-1	-2

[[
$\chi = 2$ is a sphere. See Unit B2, Fig 1.13(b). Both faces have the equation abcde = 1.

$\chi = -1$????

$\chi = -2$. This is $2\mathbb{T}$. I think the faces $abcg^{-1}f^{-1} = 1$ and $bacf^{-1}g^{-1} = 1$ are suitable.
]]

(b)(i) Since $fgf^{-1}g^{-1}c = 1$ then $c = gfg^{-1}f^{-1}$.

Substituting this value for c into the other edge equation gives
$aba^{-1}b^{-1}\, gfg^{-1}f^{-1} = 1$.

[[The surface S is $2\mathbb{T}$ so, using Unit B3 Theorem 4.1, we expect
$\chi = \chi(\mathbb{T}) + \chi(\mathbb{T}) - 2 = -2$.]]

Using the method of inserting vertices gives

$$\text{Pa Pb Pa}^{-1}\ \text{Pb}^{-1}\ \text{Pg Pf Pg}^{-1}\ \text{Pf}^{-1}\ \text{P} = 1.$$
1 3 2 1 4 7 6 5 1

The number below the vertices show the order in which they were added.
1 - a starts at P; 2 - b ends at P; 3 - a starts at P;
4 - b starts at P; 5 - g starts at P; 6 - f ends at P;
7 - f starts at P;

As the number of vertices is 1, the number of edges is 4, and the number of faces is 1 then

$$\chi = 1 - 4 + 1 = -2.$$

(b)(ii) By the Classification Theorem (Unit B3 Theorem 3.6) we know that the corresponding surface must have the same value for χ. Therefore S corresponds to the example with $j = 10$.

Question 11

(a) **b** and **d**.

[[$x_n = 1$ for infinitely many $n \in \mathbb{N}$ in **a**. $x_1 = 2 \notin \{0, 1\}$ in **c**.]]

(b)(i) When $m \le n$ then $3^{-n} \le 3^{-m}$.
Therefore $d(\mathbf{x}, \mathbf{z}) \le d(\mathbf{x}, \mathbf{y})$.
As $d(\mathbf{y}, \mathbf{z}) > 0$ then $d(\mathbf{x}, \mathbf{z}) \le d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$.

(b)(ii) **x** and **y** both agree for the first $m - 1$ terms.
As $n \le m - 1$ then both **x** and **y** first differ with **z** at the same term.
Therefore $d(\mathbf{x}, \mathbf{z}) = d(\mathbf{y}, \mathbf{z})$.
As $d(\mathbf{x}, \mathbf{y}) > 0$ then $d(\mathbf{x}, \mathbf{z}) \le d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$.

(c) In part (b) we covered all the cases where $\mathbf{x} \ne \mathbf{y}$, and $\mathbf{x} \ne \mathbf{z}$.

If $\mathbf{x} = \mathbf{z}$ then, since $d(\mathbf{x}, \mathbf{y}) \ge 0$ and $d(\mathbf{y}, \mathbf{z}) \ge 0$, we have
$$d(\mathbf{x}, \mathbf{z}) = 0 \le d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}).$$
This also covers the case when all 3 sequences are the same.

If $\mathbf{x} = \mathbf{y}$ then $d(\mathbf{x}, \mathbf{z}) = d(\mathbf{y}, \mathbf{z})$.
As $d(\mathbf{x}, \mathbf{y}) = 0$ then $d(\mathbf{x}, \mathbf{z}) \le d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$.

In all case we have $d(\mathbf{x}, \mathbf{z}) \le d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ so d satisfies (M3).

(d)(i) Let $m, n \in \mathbb{N}$ where $m,n > N \in \mathbb{N}$.

Since the sequences $\mathbf{a}_m$ and $\mathbf{a}_n$ both start with N 1s then

$$d(\mathbf{a}_m, \mathbf{a}_n) < 3^{-N}.$$

Since $((1/3)^n)$ is a basic null sequence [[A1, Sect. 1.3,]] then for any $\varepsilon > 0$ there exists a N such that $3^{-N} < \varepsilon$. [[We need $N > - \log \varepsilon / \log 3$]]
Therefore $d(\mathbf{a}_m, \mathbf{a}_n) < \varepsilon$ if $m, n > N$.

Hence $(\mathbf{a}_n)$ is a Cauchy sequence for (X, d).

(d)(ii) Since $n \geq k$ then the k^{th} item in the sequence $\mathbf{x}$ is 0 and the k^{th} item in the sequence $\mathbf{a}_n$ is 1.
Therefore the sequences $\mathbf{a}_n$ and $\mathbf{x}$ differ at or before the k^{th} term.
Hence $d(\mathbf{x}, \mathbf{a}_n) \geq 3^{-k}$.

(d)(iii) Assume that the Cauchy sequence $(\mathbf{a}_n)$ converges to an $\mathbf{x} \in X$.
Since $x_n = 1$ for only finitely $n \in \mathbb{N}$ then we can determine the value

$$k = \max \{n : x_n = 1\}.$$

From part (d)(ii) we know that $d(\mathbf{x}, \mathbf{a}_n) \geq 3^{-k}$ for all $n \geq k$.
As $d(\mathbf{x}, \mathbf{a}_n)$ is not a null sequence then the Cauchy sequence cannot converge to an $\mathbf{x} \in X$.
Therefore (X, d) is not a complete metric space.

Question 12

(a) We must show that $\mathcal{T}$ satisfies conditions (T1) - (T3) (Unit A3, Section 1).

(T1) By definition, $\phi \in \mathcal{T}$.
Since $1 \in \mathbb{N}$ then $\mathbb{N} \in \mathcal{T}$.
Hence (T1) is satisfied.

(T2) Let $U_1, U_2 \in \mathcal{T}$ and let $U = U_1 \cap U_2$. We must show that $U \in \mathcal{T}$.

If $U_1 = \phi$ or $U_2 = \phi$ then $U = \phi \in \mathcal{T}$.

Otherwise, if neither U_1 or U_2 is the empty set then
$1 \in U_1$ and $1 \in U_2$.
Hence $1 \in U$ and therefore $U \in \mathcal{T}$.

Hence (T2) is satisfied.

(T3) Let $\{ U_i : i \in I\}$ be a family of sets in $\mathcal{T}$ and let $U = \bigcup_{i \in I} U_i$.
We must show $U \in \mathcal{T}$.

If $U_i = \phi$ for each $i \in I$ then $U = \phi \in \mathcal{T}$.
Otherwise $1 \in U_j$ for some $j \in I$. Hence $1 \in U$ and therefore $U \in \mathcal{T}$.

Hence (T3) is satisfied.

As (T1) - (T3) are satisfied then $\mathcal{T}$ is a topology on $\mathbb{N}$.

(b) [[Use of Unit C1, Theorem 1.2 is an alternative method.]]

(b)(i)

[[Looking ahead to part(d) we know that $\mathbb{N}$ is path-connected. Hence it must be $\mathcal{T}$-connected. C1, Th. 4.2]]

Let U_1 and U_2 be any two non-empty open sets in $\mathcal{T}$ such that $U_1 \cup U_2 = \mathbb{N}$. Since $1 \in U_1$ and $1 \in U_2$ then their union $U_1 \cap U_2$ is not empty. Therefore there are no disconnections of $\mathbb{N}$ and so $\mathbb{N}$ is $\mathcal{T}$-connected. [[C1, Section 1.1]] (b)(ii) Let $A = \{2, 3\}$.

The subspace topology on A inherited from $\mathcal{T}$ is

$\mathcal{T}_A = \{\phi, \{2\}, \{3\}, A\}$. [[$\mathcal{T}_A = \{ U \cap A : U \in \mathcal{T}\}$]]

The sets $\{2\}$ and $\{3\}$ form a disconnection of $(A, \mathcal{T}_A)$.
Therefore $\{2, 3\}$ is not $\mathcal{T}$-connected.

[[Alternatively, as $\{2\}^c = \{3\}$ then $\{3\}$ is both open and closed. Therefore, by C1 Th. 1.2, $(A, \mathcal{T}_A)$ is not connected.]]

(c) We need to check that p is a $(\mathcal{T}(d^{(1)}), \mathcal{T})$-continuous function.

Since $(-\infty, 1/2)$ is an open set in $\mathbb{R}$ with the Euclidean topology then $[0, 1/2)$ is an open set in the Euclidean topology on $[0, 1]$.

Let U be an open subset of $\mathbb{N}$.

If $n \in U$ then $1 \in U$. So $p^{-1}(U) = [0, 1]$. This is an open subset of $[0, 1]$.
If $n \notin U$ and $1 \in U$ then $p^{-1}(U) = [0, 1/2)$. This is an open subset of $[0, 1]$.
If $n \notin U$ and $1 \notin U$ then $U = \phi$. So $p^{-1}(U) = \phi$ which is an open subset of $[0, 1]$.

As $p^{-1}(U)$ is always an open subset of $[0, 1]$ then p is a $(\mathcal{T}(d^{(1)}), \mathcal{T})$-continuous function.
Hence the function $p: [0, 1] \to \mathbb{N}$ is a path joining 1 to n.

(d) Let m, n be any elements of $\mathbb{N}$.

From part (c) we know that there are paths joining 1 to n and 1 to m.

By Unit C1 Problem 4.2 there is a path joining n to 1.

It follows, from Unit C1 Lemma 4.1, that there is a path joining n to m.

As n and m are arbitrary elements of $\mathbb{N}$ then $(\mathbb{N}, \mathcal{T})$ is path-connected.