(6) Determine which of the following series are convergent:

(i) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$; [3]

(87/15) (ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+\sqrt{n}}$; [3]

(iii) $\sum_{n=1}^{\infty} \frac{(n+1)^3}{n!}$. [4]

(7) In this question you may assume without proof that if $b > 0$ then the sequence $\left\{\frac{1}{n^b}\right\}$ is a null sequence.

(i) Let $a$ be a fixed real number with $a > 1$ and let $u_n = \frac{1}{n^a}$.

Use the integral test to prove that $\sum_{n=1}^{\infty} u_n$ is convergent. [6]

(81/21)

(ii) Let $c$ be a fixed real number with $c \leqslant 1$ and let $t_n = \frac{1}{n^c}$.

Prove that $\sum_{n=1}^{\infty} t_n$ is not convergent. [*You may assume without proof that* $\sum_{n=1}^{\infty} \frac{1}{n}$ *is not convergent.*] [4]

(8) (i) Let $a$ be a fixed real number with $|a| < \frac{1}{2}$ and let

$$u_n = \frac{(-2a)^n}{n}.$$

Prove that the series $\sum_{n=1}^{\infty} u_n$ is convergent. [3]

(ii) Let $b$ be a fixed real number with $|b| > \frac{1}{2}$ and let

(82/22)

$$u_n = \frac{(-2b)^n}{n}.$$

Prove that the series $\sum_{n=1}^{\infty} u_n$ is not convergent. [3]

(iii) Determine whether or not the series $\sum_{n=1}^{\infty} u_n$ is convergent when $u_n = \frac{(-2c)^n}{n}$ and

(*a*) $c = +\frac{1}{2}$; [2]

(*b*) $c = -\frac{1}{2}$. [2]

(9) (a) Determine whether or not the following functions are continuous at 0.

91/16

$$f(x) = \begin{cases} x^4\sin(2/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

$$g(x) = \begin{cases} x^4, & x > 0; \\ x^2+x+1, & x \leq 0. \end{cases}$$ [6]

(b) Show that the equation $x^2 + \sin^2 x = 1$ has exactly one solution in $[0, \frac{\pi}{2}]$. State clearly any results that you use. [4]

(10) (a) Determine whether or not the following functions are continuous at 0:

(i) $f(x) = \begin{cases} x^2, & x \leq 0; \\ x(2+x), & x > 0. \end{cases}$

(90/16)

(ii) $f(x) = \begin{cases} -x^2, & x \leq 0; \\ 1/x^2, & x > 0. \end{cases}$ [6]

(b) Prove that the following sequence is convergent, and determine its limit:

$$\{\cos(e^{1/n} - 1)\}.$$ [4]