ANALYSIS B

1 (a) Prove that the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} x^2 \text{ if } x < 0 \\ x^3 \text{ if } x \geq 0 \end{cases}$$

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is differentiable at every point of $\mathbb{R}$.
State, without proof, whether the function f' is differentiable at 0.
(A rough sketch of the graph of f' near the origin may help.) [4]

(b) Prove that

$$\lim_{x \to 0} \frac{\sin(2x)\sinh x}{\cosh x - 1}$$

$2\cos(2x)\sinh x + \sin(2x)\cosh x$
$\sinh x$ | 0

exists, and determine its value. [6]

2 (a) (i) Draw a rough sketch of the graph of the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} 2x^3, & \text{if } x \leq 0, \\ 3x^2, & \text{if } x > 0. \end{cases}$$

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Determine at which points of $\mathbb{R}$ the function f is differentiable. [4]

(ii) Draw a rough sketch of the graph of the function f', where f is the function defined in part (i). State without proof whether f' is differentiable at 0. [2]

(b) Prove the following inequality:

$$10x^{\frac{1}{10}} \leq x + 9, \qquad \text{for } x \in [0, 1].$$ [4]

3 (a) Determine the Taylor polynomial of degree 3 for the function

$$f(x) = \log_e(1 + x)$$

at the point $-\frac{1}{2}$. [3]

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(b) Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n} x^n.$$ [3]

(c) Using basic power series, or otherwise, determine the Taylor polynomial at 0 of degree 4 for the function

$$f(x) = \cos(e^x - 1).$$ [4]

4 (a) (i) Determine the Taylor polynomial $T_3(x)$ for the function $f(x) = (1 + 2x)^{-\frac{1}{2}}$ at 0. [3]

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(ii) Show that $T_3(x)$ approximates $f(x)$ to within 10^{-3} on the interval $[0, \frac{1}{10}]$. [3]

(b) Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty} (n^3 + 3^{-n})(x + 1)^n.$$ [4]

5 (i) Prove that

$$\frac{\pi}{4} \leq \int_0^{\pi/2} \frac{dx}{(1 + 3\sin^2 x)^{\frac{1}{2}}} \leq \frac{\pi}{2}.$$ [5]

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(ii) Determine a number λ such that

$$\frac{(2n)!}{(n!)^2} \sim \lambda \frac{2^{2n}}{n^{\frac{1}{2}}} \qquad \text{as } n \longrightarrow \infty.$$ [5]