(4) This question concerns the following group G of matrices under matrix multiplication:

$$G = \left\{ \begin{pmatrix} p & 0 \\ a & 1 \end{pmatrix} : p, a \in \mathbb{R}, p \neq 0 \right\}$$

and the subset

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} : a \in \mathbb{R} \right\}.$$

(i) Calculate the following product of two elements in G [1]

$$\begin{pmatrix} p & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} q & 0 \\ b & 1 \end{pmatrix}$$

(82/15)

(ii) Let ϕ be the function from G to the group $(\mathbb{R}^*, \times)$ of non-zero reals under multiplication given by

$$\phi : \begin{pmatrix} p & 0 \\ a & 1 \end{pmatrix} \longmapsto p.$$

Prove that ϕ is a homomorphism. [3]

(iii) Hence, or otherwise, prove that H is a normal subgroup of G and that $G/H \cong (\mathbb{R}^*, \times)$. [6]

(5) Consider the set

$$G = \left\{ A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a \in \mathbb{R} - \{0\}, b \in \mathbb{R} \right\}.$$

(i) Show that G is a group under matrix multiplication. [3]

(ii) Show that the function

$$\phi : G \longrightarrow (\mathbb{R}, +)$$
$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \longmapsto b/a$$

(87/17) & (93/17)

is a homomorphism. [3]

(iii) Show that

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R} - \{0\} \right\}$$

is a normal subgroup of G, and explain why

$$G/H \cong (\mathbb{R}, +).$$ [4]

(6) In this question M denotes the set of all non-singular 2×2 matrices, and ϕ denotes the function

$$\phi : M \longrightarrow (\mathbb{R} - \{0\}, \times)$$
$$A \longmapsto (\det(A))^2.$$

You may assume that the determinant function is a homomorphism.

(a) (i) Prove that ϕ is a homomorphism.

(ii) Find H, the kernel of ϕ. [4]

(90/17)

(b) Determine the coset AH where

$$A = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}.$$

Specify the set of cosets of H in M. [4]

(c) Explain why the quotient group M/H exists and identify this group up to isomorphism. [2]

(7)

Let G be a group, and $\theta : G \to G$ the function given by $\theta(g) = g^{-1}$.

(85/15)

(i) Prove that θ is surjective (i.e. onto).

(ii) Prove that θ is injective (i.e. one-one).

(iii) Prove that θ is a homomorphism if, and only if, G is abelian.

(8)

Let G be a group and $\theta : G \longrightarrow G$ a function given by $\theta(x) = x^2$.

(83/15)

(i) Prove that θ is a homomorphism if and only if G is abelian. [6]

(ii) If $G = \mathbb{Z}_6$, the cyclic group of order 6, show that θ is *not* injective (i.e. θ is not one-one). [4]