## (C) Vector Spaces

(15) (a) Show that the set

$$S = \{(1,2,3), (-1,2,1), (-1,10,9)\}$$

is a linearly dependent set but that $\{(1,2,3), (-1,2,1)\}$ is a linearly independent set. (93)

(b) The subspace $T$ of $\mathbb{R}^3$ is defined by

$$T = \{(x,y,z) \in \mathbb{R}^3 : x + y - z = 0\}.$$

(You are NOT asked to show that $T$ is a subspace.)
Prove that $\{(1,2,3), (-1,2,1)\}$ is a basis for the subspace $T$.

(c) Find an orthogonal basis for $T$ including the vector (1,2,3). [7]

(16) Let $S = \{(1,1,0), (1,3,-2), (1,2,-1)\}$ be a set of three vectors in $\mathbb{R}^3$, and let (89)

$$T = \{(x,y,z) \in \mathbb{R}^3 : x - y - z = 0\}.$$

You may *assume* that $T$ is a subspace of $\mathbb{R}^3$.

(a) Show that $S$ is a linearly dependent set, but that $\{(1,1,0), (1,3,-2)\}$ is a linearly independent set. [3]

(b) Show that $(1,1,0)$ and $(1,3,-2) \in T$, but that $(1,2,3) \notin T$. [1]

(c) Hence, or otherwise, show that $\langle S \rangle = T$. [2]

(17) Let $S = \{(u, v, u + w, w - v) : u, v, w \in \mathbb{R}\}$ be a subset of $\mathbb{R}^4$.

(i) Show that $S$ is a subspace, by showing that $S$ is closed under addition and closed under scalar multiplication.

(ii) Show that the set $\{(1, 1, 1, -1), (0, 1, 1, 0), (1, 0, 2, 1)\}$ is linearly independent *and* spans $S$.

(iii) Write down, giving brief reasons for your answer, the dimension of $S$. (84)

(18) Let $S = \{(a, b, a, -2b) : a \in \mathbb{R}, b \in \mathbb{R}\}$ be a subset of $\mathbb{R}^4$.

(i) Show that $S$ is a subspace of $\mathbb{R}^4$. [2]

(ii) Show that each element of $S$ can be expressed as a unique linear combination of the vectors $(1, 0, 1, 0)$ and $(0, 1, 0, -2)$. [2]

(iii) What is the dimension of $S$? Justify your answer. [1]

(iv) Write down a basis for $\mathbb{R}^4$ which includes the two vectors $(1, 0, 1, 0)$ and $(0, 1, 0, -2)$. [2] (81)

(19) Let $S = \{(0,1,2), (1,1,1), (3,1,-1)\}$ be a set of three vectors in $\mathbb{R}^3$, and let

$$T = \{(x,y,z) \in \mathbb{R}^3 : x - 2y + z = 0\}.$$ (93)

You may assume that $T$ is a subspace of $\mathbb{R}^3$.

(a) Show that $S$ is a linearly dependent set, but that $\{(0,1,2), (1,1,1)\}$ is a linearly independent set.

(b) Prove that $\{(0,1,2), (1,1,1)\}$ is a basis for $T$.

(c) Hence or otherwise show that $\langle S \rangle = T$. [7]

(20) (i) Show that $\{(1, 2, 3), (3, 2, 1), (0, 1, 1)\}$ is a linearly independent set of vectors in $\mathbb{R}^3$. [3]

(ii) Express $(0, 0, -2)$ as a linear combination of the above set of vectors. [2]

(iii) Write down the dimension of the subspace of $\mathbb{R}^3$ spanned by $\{(1, 2, 3), (3, 2, 1), (0, 1, 1), (0, 0, -2)\}$. [1]

Justify your answer. [1] (80)