LINEAR ALGEBRA

1 (81/13)

Let $A = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$, where a and b are given elements of $\mathbb{R}$.

(i) Find the eigenvalues of A. [2]

(ii) Suppose that $a \neq b$. Explain briefly how to find an invertible matrix P such that

$$P^{-1}AP = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$ [4]

(iii) Prove that if $B = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, where $a \in \mathbb{R}$, then there is no invertible matrix P such that $P^{-1}BP$ is a diagonal matrix. [4]

2 (82/13)

The matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ represents a linear transformation $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with respect to the standard basis in both the domain and codomain.

(i) Determine the characteristic equation of A and by solving this equation show that the eigenvalues of f are 0, 1 and 2. [4]

(ii) Find bases for the eigenspaces of f corresponding to the eigenvalues of f. [3]

(iii) Hence find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of f. [1]

(iv) Write down an orthogonal matrix P such that P^TAP is diagonal, and write down this diagonal matrix. [2]

3 (80/13)

The matrix $A = \begin{pmatrix} 5 & 2 & 2 \\ 2 & 2 & -4 \\ 2 & -4 & 2 \end{pmatrix}$ represents the linear transformation $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with respect to the standard basis in both the domain and codomain.

The eigenvalues of A are -3 and 6, which is a repeated root of the characteristic equation of A.

(i) Find bases for the eigenspaces of f corresponding to the eigenvalues given above.

(ii) Hence obtain an orthonormal basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ for $\mathbb{R}^3$ consisting of eigenvectors of f.

(iii) If the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of part (ii) are the columns of a matrix P, write down the entries in the matrix $B = P^TAP$.

(iv) What is the relationship between the matrix B and the linear transformation f?

4 (89/13)?

The matrix $A = \begin{pmatrix} 4 & 2 & 2 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ represents a linear transformation $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with respect to the standard basis in both domain and codomain.

(a) Determine the eigenvalues of f. [3]

(b) Determine the eigenspace corresponding to each eigenvalue. [3]

(c) Write down a basis for $\mathbb{R}^3$ consisting of eigenvectors of f. [2]

(d) A vector in the domain has coordinates $(1, -2, 3)$ with respect to the standard basis. Find the coordinates of this vector with respect to the basis of part (c), and so find the coordinates of its image under f, also with respect to the basis of part (c). [2]

5 (93/13)?

The matrix $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}$ represents the linear transformation $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with respect to the standard basis in both the domain and the codomain.

(a) Determine the characteristic equation of A and solve this equation to find the eigenvalues of f. [3]

(b) Find bases for the eigenspaces of f. [4]

(c) The matrix, C, of f with respect to the eigenvector basis in domain and codomain can be expressed as $C = P^TAP$. Find the matrix P and write down the matrix C. [3]