6 (i) Let

$$I_n = \int_1^e x^2(\log_e x)^n\,dx, \qquad n = 0, 1, 2, \ldots.$$

(a) Evaluate I_0.

(b) Prove that

$$I_n = \frac{1}{3}e^3 - \frac{1}{3}nI_{n-1}, \quad \text{for } n = 1, 2, \ldots.$$

87/21) (c) Deduce the values of I_1 and I_2. [5]

(ii) Show that

$$\int \frac{dx}{x(\log_e x)^{1/2}} = 2(\log_e x)^{1/2},$$

and deduce by the Integral Test that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log_e n)^{1/2}}$$

is divergent. [5]

7 (a) Use Stirling's Formula

(i) to estimate the following number to 1 significant figure:

$$1000\sqrt{2\pi}\,\frac{400!}{4^{400}(100!)^4};$$ [3]

(ii) to determine a real number λ such that

$$\frac{(2n)!}{(n!)^2} \sim \lambda\frac{2^{2n}}{\sqrt{n}} \text{ as } n \to \infty.$$ [2]

(89/22) (b) Let f be the function

$$f : [0, 2] \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} x, & x \in [0, 1[\\ x - 1, & x \in [1, 2] \end{cases}.$$

(i) Sketch the graph of f. [1]

(ii) Determine the values of the Riemann sums $L(f, P)$ and $U(f, P)$ for the partition P of $[0, 2]$, where

$$P = \left\{[0, \tfrac{2}{3}], [\tfrac{2}{3}, \tfrac{3}{2}], [\tfrac{3}{2}, 2]\right\}.$$ [4]

8 This question concerns the linear flow with velocity function

$$V(x, y) = (x + 4y, x - 2y).$$

(i) Write down

(a) the matrix $\mathbf{A}$ of the flow; [1]

(88/22) (b) the first order differential equations satisfied by the coordinate functions f, g of any flow function for this flow; [1]

(c) a second order differential equation satisfied by these coordinate functions. [1]

(ii) Find the general solution of the differential equation in part (i)(c). [2]

(iii) Determine the flow function α corresponding to V which satisfies $\alpha(0) = (5, 0)$. [5]

9 This question concerns the linear flow with velocity function

$$V(x, y) = (x + 2y, 2x + y).$$

(i) Write down

(a) the matrix $\mathbf{A}$ of the flow; [1]

(87/22) (b) the first order differential equations satisfied by the coordinate functions f, g of any flow function for this flow; [1]

(c) a second order differential equation satisfied by these coordinate functions. [1]

(ii) Find the general solution of the differential equation in part (i)(c). [2]

(iii) Determine the flow function α corresponding to V which satisfies $\alpha(0) = (2, 0)$. [5]