(15)

In this question, we investigate the real vector space V of all polynomials in x of degree 2 or less, i.e. all polynomials of the form

$$p(x) = ax^2 + bx + c \quad (a, b, c \in \mathbb{R}).$$

The operations of addition and scalar multiplication are those of polynomials.
You are **not** asked to verify that V is a vector space.

(84/14)

(*a*) Determine if the following two subsets S_1, S_2 of V are subspaces of V.

(i) $S_1 = \{p(x) \in V: p(\sqrt{2}) = 0\}$.

(ii) $S_2 = \{p(x) \in V: p(x) = (x-1)(x-d) \text{ for some } d \in \mathbb{R}\}$. [5]

(*b*) Show that the function

$t: V \longrightarrow V$

$p(x) \longmapsto p(0) + x^2p(1)$

is a linear transformation, and determine the kernel of t in the form

$$\{ax^2 + bx + c: \text{some condition on } a, b, c\}.$$ [5]

(16)

In this question we investigate the vector space V of all functions with domain and codomain $\mathbb{R}$. Addition in V is defined by

$(f+g)(x) = f(x) + g(x)$ for all x in $\mathbb{R}$,

and scalar multiplication by $\lambda \in \mathbb{R}$ is defined by

$(\lambda f)(x) = \lambda \cdot f(x)$ for all x in $\mathbb{R}$.

You are **not** *asked to verify that V is a vector space.*

(84/14)

(i) Determine if the following two subsets S_1, S_2 of V are subspaces of V. [4]

(*a*) S_1 is the set of continuous functions $\mathbb{R} \longrightarrow \mathbb{R}$.

(*b*) $S_2 = \{f \in V: f(3) = 1\}$.

(ii) Show that the function $t: V \longrightarrow \mathbb{R}$

$f \longmapsto f(1) - f(0)$ [6]

is a linear transformation, and determine the kernel of t.

(17)

In this question we investigate the vector space V of all functions $f: \mathbb{R} \longrightarrow \mathbb{R}$ in which addition of f, g in V is defined by

$(f+g)(x) = f(x) + g(x)$ for all x in $\mathbb{R}$

and scalar multiplication by $\alpha \in \mathbb{R}$ is defined by

$(\alpha f)(x) = \alpha f(x)$ for all x in $\mathbb{R}$.

Let $S = \{f \in V: f(0) = 0\}$.

(i) Prove that S is a subspace of V. [3]

(ii) Given any function f in V we can define a function f_0 in V by

$f_0(x) = f(x) - f(0)$ for all x in $\mathbb{R}$.

(82/14)

Let the function $t: V \longrightarrow V$ be defined by $t(f) = f_0$.

(*a*) Prove that t is a linear transformation. [3]

(*b*) Prove that $\mathrm{Ker}(t)$ is the set of constant functions in V. [2]

(*c*) Prove that $t(f) = f$ if and only if $f \in S$. [2]

(18)

Consider the vector space of functions

$$V = \{f(x) = \alpha + \beta\cos x + \gamma\sin x : \alpha, \beta, \gamma \in \mathbb{R}\}.$$

(You are *not* asked to verify that V is a vector space.)

(i) Determine whether each of the following sets is a subspace of V:

(88/14)

(*a*) $S = \{f(x) = \alpha + \beta\cos x + \beta\sin x : \alpha, \beta \in \mathbb{R}\}$;

(*b*) $T = \{f(x) = \alpha + \beta\cos x + \gamma\sin x : \alpha, \beta, \gamma \in \mathbb{R}, \alpha \neq 0\}$. [5]

(ii) Show that the function

$t: V \to \mathbb{R}$

$f(x) \mapsto f(0) + f(3\pi/2)$

is a linear transformation and determine the kernel of t in the form

$$\{f(x) = \alpha + \beta\cos x + \gamma\sin x: \text{ some condition on } \alpha, \beta, \gamma\}.$$ [5]