(6) (94/13)

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 3 & 1 \\ 1 & 5 & -3 \end{pmatrix}$ represents a linear transformation $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with respect to the standard basis in both the domain and the codomain.

(a) Find the row-reduced form of the matrix $\mathbf{A}$. [2]

(b) Hence determine the kernel of f. [2]

(c) Determine the dimension of $\mathrm{Im}(f)$, and find a basis for $\mathrm{Im}(f)$. [3]

(d) Determine whether the vector $\mathbf{v} = (2, 3, 4)$ belongs to $\mathrm{Im}(f)$. [3]

(7) (88/13)

Repeat (6) for $\mathrm{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & -1 & -3 \end{pmatrix}$.

(8) (81/14)

This question concerns the vector space $\mathbb{R}^\infty$ of all infinite sequences $(a_1, a_2, a_3, \ldots, a_n, \ldots)$ of real numbers and the following function

$$t: \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$$

$$(a_1, a_2, a_3, \ldots, a_n, \ldots) \longmapsto (a_1 + a_2, a_2 + a_3, a_3 + a_4, \ldots, a_n + a_{n+1}, \ldots).$$

(i) Prove that t is a linear transformation. [3]

(ii) Find a basis for $\mathrm{Ker}(t)$ and write down $\dim(\mathrm{Ker}(t))$. [3]

?

(iii) Show that $(2, 4, 8, \ldots, 2^n, \ldots)$ is an eigenvector of t and write down the corresponding eigenvalue. [3]

(iv) Write down an eigenvector with eigenvalue 10. [1]

(9) (91/14)

This question is about the vector space of functions

$$V = \{(a + bx + cx^2)e^{2x} : a, b, c \in \mathbb{R}\}$$

and the linear transformation

$$L : V \longrightarrow V$$
$$f \longmapsto f'$$

(i.e. L is the differentiation operator). You are NOT asked to prove either that V is a vector space or that L is a linear transformation.

(a) Show that L is one-one. [2]

(b) Find the matrix of L with respect to the basis $\{e^{2x}, xe^{2x}, x^2e^{2x}\}$ in both domain and codomain. [2]

(c) Find any eigenvalues of L and the corresponding eigenspaces. [2]

(d) Determine the inverse transformation L^{-1}, giving your answer in the form

$$L^{-1} : (a + bx + cx^2)e^{2x} \longmapsto \ldots$$ [4]

(10) (91/13)

This question concerns the set

$$S = \{(a, b, 2a - 3b) : a, b \in \mathbb{R}\}.$$

(a) Prove that S is a subspace of $\mathbb{R}^3$. [3]

(b) Determine a basis for S and write down the dimension of S. [3]

(c) Determine two vectors in S that are orthogonal. [2]

(d) Find an orthogonal basis for $\mathbb{R}^3$ that includes your solution to part (c) above. [2]

(10') (95/16)

This question concerns the set

$$S = \{(a, b, a - b, b - 2a) : a, b \in \mathbb{R}\}.$$

(a) Prove that S is a subspace of $\mathbb{R}^4$. [3]

(b) Show that $\{(1, 1, 0, -1), (2, 1, 1, -3)\}$ is a basis for S, and state the dimension of S. [3]

(c) Find an orthogonal basis for S that includes the vector $(1, 1, 0, -1)$. [2]

(d) Express the vector $(2, 1, 1, -3)$ in terms of your orthogonal basis from part (c). [2]