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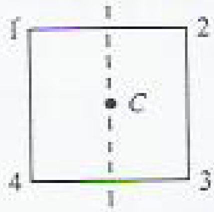
This question concerns the symmetry group G of the square.



1 2 3 4
2 1 2 1
3 4 1 2
4 3 4 3

Let $h \in G$ be the clockwise rotation through $\pi/2$ about the centre C, and let $g \in G$ be the reflection about the dotted line shown on the figure.

(i) Write h and g in cycle form, using the numbering of the location of the vertices as shown. $h(1234)$ $g=(12)(34)$ [2]

(ii) Write down the conjugate ghg^{-1} of h and state its order. (1432) 4 [2]

(iii) Write down a brief reason why g and h are not conjugate. [1]

(iv) Let ϕ be a homomorphism from G to a group X. Use part (ii) to show that $\phi(h)$ and $\phi(h^{-1})$ are conjugates in X, where h is the element of G referred to above. h, h^{-1} conjugate in G, conjugate in X. [2] (83)

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Let g, h and k be the following three elements of the symmetric group S_5:

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix};\ h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 3 & 1 \end{pmatrix};\ k = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 2 & 3 & 1 \end{pmatrix}.$$

(i) Express g, h and k in cycle form. $(124)(35)$ $h(14325)$ $k(15)(24)$ [2]

(ii) Give a brief reason why g and k are conjugate in S_5 but h is not conjugate in S_5 to either of the other two. [1]

(iii) Write down an element x in S_5 such that $xgx^{-1} = k$. [2]

(iv) Write down a homomorphism $\phi \colon S_5 \longrightarrow \mathbb{Z}_2$ such that $h \in \operatorname{Ker}(\phi)$ but $g, k \notin \operatorname{Ker}(\phi)$. even perm $\to 0$ odds $\to 1$ [2] (81)

(89)

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This question concerns the group G of diagonal matrices with real positive entries

$$G = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R}, a > 0, b > 0 \right\},$$

under matrix multiplication, and the group $(\mathbb{R} - \{0\}, \times)$ of non-zero real numbers under multiplication. You may *assume* that both of these are groups.

(a) Show that G is Abelian. $\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d\end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd\end{pmatrix}$ [1]

(b) Show that the function $\begin{pmatrix} c & 0 \\ 0 & d\end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd\end{pmatrix}$

$$f : G \longrightarrow \mathbb{R} - \{0\}$$
$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \longmapsto a$$

$\phi\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix} \times \phi\begin{pmatrix} c & 0 \\ 0 & d\end{pmatrix} = \phi\left(\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d\end{pmatrix}\right)$

is a homomorphism. $ac = ac$ [2]

(c) Determine the kernel of the homomorphism f in part (b). [2]

$\begin{pmatrix} 1 & 0 \\ 0 & b\end{pmatrix}$

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The mapping

(91)

$$\theta : (\mathbb{Z}_8, +_8) \longrightarrow (\mathbb{Z}_4, +_4)$$

is a homomorphism such that $\theta(1) = 2$.

(a) Write down an equation relating $\theta(m)$, $\theta(n)$ and $\theta(m +_8 n)$, where $m, n \in \mathbb{Z}_8$.

(b) Write down the image of each element of $\mathbb{Z}_8$ under θ.

(c) Find $\operatorname{Ker}(\theta)$. [5]

0 1 2 3 4 5 6 7
1 2 4 3 1 2 4 3

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Let $\phi : (\mathbb{Z}_8, +_8) \longrightarrow (\mathbb{Z}_5^*, \times_5)$ be the homomorphism given by $\phi(u) = 2^u \pmod 5$.

(a) Find the image under ϕ of each element of $\mathbb{Z}_8$. $\operatorname{Im}\phi = (1234)$

(b) Find $\operatorname{Ker}(\phi)$ and $\operatorname{Im}(\phi)$ and the cosets of $\operatorname{Ker}(\phi)$ in $\mathbb{Z}_8$. [5]

$\operatorname{Ker}(\phi) = (0, 4)$ cosets $(15), (26)(37)$ (93)