9 This question concerns the subset X of $\mathbb{R}^2$ where

$X = \{(a,b) : b \neq 0\}$ (that is, $\mathbb{R}^2$ excluding the x-axis)

and the binary operation $*$ defined by

$(a,b) * (c,d) = (ad + bc, bd)$.

(a) Verify that the element $(0,1)$ acts as the identity element in X. [1]

(b) Find a formula for the inverse of the element $(a,b) \in X$, and hence show that $(X,*)$ satisfies the inverses axiom G3. [2]

In parts (c), (d) and (e) below you may assume that $(X,*)$ is a group: you are NOT asked to check the remaining group axioms G1 and G4.

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(c) Prove that

$$\phi : (X,*) \longrightarrow (\mathbb{R},+)$$
$$\phi : (a,b) \longmapsto a/b$$

is a homomorphism. [2]

(d) Find the kernel of the homomorphism ϕ defined in part (c). [2]

(e) Justify the statement that the quotient group

$X/\{(0,b) : b \in \mathbb{R}, b \neq 0\}$ order

exists, and show that it is isomorphic to $(\mathbb{R},+)$. [3]

GROUP ACTIONS

1 Let S_4 denote the group of all permutations of the symbols $\{1, 2, 3, 4\}$. S_4 acts on *itself* as follows: if $\rho, \sigma \in S_4$ then

$\rho \cdot \sigma = \rho\sigma\rho^{-1}$. $\rho \cdot \sigma$

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(i) Show that this satisfies the two axioms for a group action. [2]

(ii) Determine the orbit of (12) under this action. $\mathrm{Stab}(\rho(12))$ [3]

(iii) Determine the stabilizer of (12). [3]

(iv) Write down a brief description of the set of all orbits under this action. [2]

S_4

2 Let G be the symmetry group of the rhombus below (a parallelogram whose diagonals are perpendicular). The corners of the rhombus are $C1$, $C2$, $C3$, $C4$ and its sides are $S1$, $S2$, $S3$, $S4$. Its diagonals are $D1$ and $D2$.

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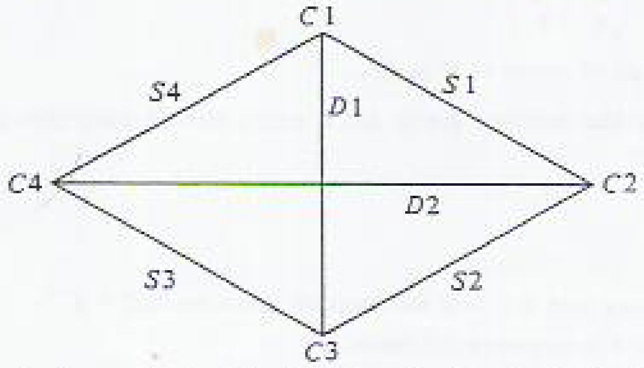

Thus $G = \{e, h, v, \rho\}$, where e is the identity, h is reflection in the horizontal diagonal, v is the reflection in the vertical diagonal, and ρ is rotation by π about the centre.

The group G defines a group action on the set

$X = \{C1, C2, C3, C4, S1, S2, S3, S4, D1, D2\}$.

(You are *not* asked to prove this.)

(i) Write down all the orbits under the action of G on X. [3]

(ii) Write down the stabilizers of $C1$, $S1$ and $D1$. [3]

(iii) Explain why it is impossible to construct a group action using G in which some orbit has three