(c) $$f:\begin{pmatrix}2\\-1\end{pmatrix}\longmapsto\begin{pmatrix}-5\\-1\end{pmatrix}=\alpha\begin{pmatrix}2\\-1\end{pmatrix}+\beta\begin{pmatrix}3\\2\end{pmatrix},$$ 1M for correct use of codomain basis (We must express the images in terms of the new codomain basis.)

where
$$\left.\begin{array}{r}2\alpha+3\beta=-5\\-\alpha+2\beta=-1\end{array}\right\}\implies\alpha=-1,\ \beta=-1.$$

Also,
$$f:\begin{pmatrix}3\\2\end{pmatrix}\longmapsto\begin{pmatrix}-4\\9\end{pmatrix}=\alpha\begin{pmatrix}2\\-1\end{pmatrix}+\beta\begin{pmatrix}3\\2\end{pmatrix},$$
where
$$\left.\begin{array}{r}2\alpha+3\beta=-4\\-\alpha+2\beta=9\end{array}\right\}\implies\alpha=-5,\ \beta=2.$$

Hence the matrix is $\begin{pmatrix}-1&-5\\-1&2\end{pmatrix}$. 1A

Question 4

(a) The characteristic equation of $\mathbf{A}$ is
$$\det(\mathbf{A}-\lambda\mathbf{I})=0,$$ $\frac{1}{2}$M for the equation

i.e.
$$\begin{aligned}0&=\begin{vmatrix}1-\lambda&-2&2\\0&-3-\lambda&4\\0&-2&3-\lambda\end{vmatrix}\\&=(1-\lambda)\begin{vmatrix}-3-\lambda&4\\-2&3-\lambda\end{vmatrix}+2\begin{vmatrix}0&4\\0&3-\lambda\end{vmatrix}+2\begin{vmatrix}0&-3-\lambda\\0&-2\end{vmatrix}\\&=(1-\lambda)(-(3+\lambda)(3-\lambda)+8)\\&=(1-\lambda)(\lambda^2-1)\\&=(1-\lambda)(\lambda-1)(\lambda+1).\end{aligned}$$ $\frac{1}{2}$M

So the eigenvalues of $\mathbf{A}$ are 1, 1 and -1. $\frac{1}{2}$A

(b) The eigenvector equations $\mathbf{A}\mathbf{v}=\lambda\mathbf{v}$, where $\mathbf{v}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, are
$$\begin{aligned}(1-\lambda)x\quad -2y+\quad 2z&=0,\\(-3-\lambda)y+\quad 4z&=0,\\-2y+(3-\lambda)z&=0.\end{aligned}$$ $\frac{1}{2}$M

When $\lambda=1$, the eigenvector equations become
$$-2y+2z=0,$$
$$-4y+4z=0,$$
$$-2y+2z=0,$$
so that $y=z$. Thus two linearly independent eigenvectors of $\mathbf{A}$ are, for example, $\frac{1}{2}$A for the equation
$$\mathbf{v}_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\qquad\mathbf{v}_2=\begin{pmatrix}0\\1\\1\end{pmatrix}.$$ $\frac{1}{2}$A, $\frac{1}{2}$A

When $\lambda=-1$, the eigenvector equations become
$$2x-2y+2z=0,$$
$$-2y+4z=0,$$
$$-2y+4z=0.$$
It follows that
$$y=2z\quad\text{and}\quad x=y-z=z.$$ $\frac{1}{2}$A for the equations

So a corresponding eigenvector of $\mathbf{A}$ is
$$\mathbf{v}_3=\begin{pmatrix}1\\2\\1\end{pmatrix}.$$ $\frac{1}{2}$A

Hence we may take the set $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ as a basis for $\mathbb{R}^3$ consisting of eigenvectors of $\mathbf{A}$. $\frac{1}{2}$A for the conclusion