**Question 5**

We shall show that $\sup E = 1$. 1A (This fact is needed in what follows.)

Since each element of $E$ is of the form $1 - \dfrac{2}{n^2}$, $n \in \mathbb{N}$, and

$$1 - \frac{2}{n^2} \le 1,$$

1 is *an* upper bound of $E$. $\frac{1}{2}$M

Suppose that $m' < 1$. Then $\frac{1}{2}$M for correct hypothesis

$$\begin{aligned} 1 - \frac{2}{n^2} > m' &\iff 0 < \frac{2}{n^2} < 1 - m' \\ &\iff \frac{n^2}{2} > \frac{1}{1-m'} > 0 \\ &\iff n > \sqrt{\frac{2}{1-m'}}. \end{aligned}$$

1A for inequalities

By the Archimedean Property of $\mathbb{R}$, there is an integer $N$ such that

$$N > \sqrt{\frac{2}{1-m'}}.$$

$\frac{1}{2}$M for use of Archimedean Property

Hence there is an element of $E$ greater than $m'$. Hence 1 is the *least* upper bound of $E$. $\frac{1}{2}$M

**Question 6**

(a) Dividing through by the dominant term $n!$, we obtain

$$a_n = \frac{n! + n^2}{3(n!) - n^3} = \frac{1 + n^2/n!}{3 - n^3/n!}.$$

1M for dividing by $n!$

Since $\{n^2/n!\}$ and $\{n^3/n!\}$ are basic null sequences, we deduce by the Combination Rules that 1M

$$\begin{aligned} \lim_{n\to\infty} a_n &= \frac{1+0}{3-0} \\ &= \tfrac{1}{3}. \end{aligned}$$

1A

(b) Since $\{6^n/n!\}$ is a basic null sequence with positive terms, $\{n!/6^n\}$ tends to infinity, by the Reciprocal Rule. 1A $\frac{1}{2}$M for mentioning the relevant rule

Hence $\{n!/6^n\}$ is unbounded, $\frac{1}{2}$M
and so is not convergent. 1A

**Question 7**

(a) Since

$$0 \le \sin(1/n) \le 1, \qquad \text{for } n = 1, 2, \ldots,$$

we have

$$0 \le \frac{\sin(1/n)}{n^3} \le \frac{1}{n^3}, \qquad \text{for } n = 1, 2, \ldots.$$

$\frac{1}{2}$A

Hence

$$\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^3} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{n^3},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{is a basic convergent series.}$$

$\frac{1}{2}$A

Thus, by the Comparison Test, 1M

$$\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^3} \quad \text{is convergent.}$$

1A