.5

(a) The elements of G are

the identity	e,		
reflection in a vertical axis	$(26)(35)$,		
reflection in a horizontal axis	$(14)(23)(56)$,	$1\frac{1}{2}$A	($\frac{1}{2}$ off for each error or extra element)
rotation through π about centre	$(14)(25)(36)$.		

The order of G is 4. $\frac{1}{2}$A

(b) The subgroups are $\{e\}$, $\{e,(26)(35)\}$, $\{e,(14)(23)(56)\}$, $\{e,(14)(25)(36)\}$, G. 2A ($\frac{1}{2}$ off for each omission)

(c) G is not cyclic as each of its elements (other than e) has order 2. (That is, the group G does not contain a generator.) $\frac{1}{2}$A; $\frac{1}{2}$ for reason

(d) G is isomorphic to the symmetry group of the rectangle, $S(\square)$. 1A for correct group

$$\begin{array}{ccc} G & \longrightarrow & S(\square) \\ e & \longmapsto & e \\ (14)(25)(36) & \longmapsto & a \\ (26)(35) & \longmapsto & r \\ (14)(23)(56) & \longmapsto & s \end{array}$$

2 There are other possible isomorphisms. Anything that maps identity to identity and is one-one will do.

(e) By Lagrange's Theorem, subgroups of $S(\triangle)$ must have order which divides 6. Thus a subgroup of $S(\triangle)$ must have order 1, 2, 3 or 6. But the order of G is 4, which is not one of these. Hence $S(\triangle)$ can have no subgroup of order 4 and so no subgroup of $S(\triangle)$ can be isomorphic to G.

$\frac{1}{2}$M for Lagrange's Theorem
$\frac{1}{2}$A for order of $S(\triangle)$
$\frac{1}{2}$M for reasoning
$\frac{1}{2}$A for conclusion

Question 16

(a) S is non-empty because, for example, $(0,0,0) \in S$. $\frac{1}{2}$M

We must show that S is closed under vector addition and scalar multiplication.

Let $\mathbf{u}_1 = (a_1, b_1, -a_1 - b_1) \in S$ and $\mathbf{u}_2 = (a_2, b_2, -a_2 - b_2) \in S$. Then $\frac{1}{2}$M

$$\begin{aligned} \mathbf{u}_1 + \mathbf{u}_2 &= (a_1, b_1, -a_1 - b_1) + (a_2, b_2, -a_2 - b_2) \\ &= (a_1 + a_2, b_1 + b_2, -(a_1 + a_2) - (b_1 + b_2)) \end{aligned}$$

has the correct form and so belongs to S. $\frac{1}{2}$A

Similarly,

$$\begin{aligned} \lambda\mathbf{u}_1 &= \lambda(a_1, b_1, -a_1 - b_1) \\ &= (\lambda a_1, \lambda b_1, -\lambda a_1 - \lambda b_1) \end{aligned}$$ $\frac{1}{2}$A

has the correct form and so belongs to S.

(b) Both $(1,1,-2) = (1,1,-1-1)$ and $(-2,1,1) = (-2,1,2-1)$ belong to S since they are of the correct form. $\frac{1}{2}$M

These two vectors are linearly independent, since they are not parallel. $\frac{1}{2}$A $\frac{1}{2}$ for reason

These two vectors span S, since we can solve the equation

$$\alpha(1,1,-2) + \beta(-2,1,1) = (a, b, -a-b)$$ $\frac{1}{2}$M

to give

$$\begin{cases} \alpha - 2\beta = a, \\ \alpha + \beta = b, \\ -2\alpha + \beta = -a - b. \end{cases}$$ 1A

Solving the first two equations gives $\alpha = \frac{1}{3}a + \frac{2}{3}b$, $\beta = \frac{1}{3}b - \frac{1}{3}a$, and substituting these solutions into the third equation gives

$$-2\alpha + \beta = -\tfrac{2}{3}a - \tfrac{4}{3}b + \tfrac{1}{3}b - \tfrac{1}{3}a = -a - b,$$

as required. So these values of α and β satisfy all three equations.

Thus $\{(1,1,-2),(-2,1,1)\}$ is a basis for S, and so S has dimension 2. 1A