Question 17

Determine which of the following functions f are continuous at 0.

(a) $f(x) = \begin{cases} x^3, & x \leq 0 \\ -x^2, & x > 0 \end{cases}$ [3]

(b) $f(x) = \begin{cases} x^3, & x \leq 0 \\ x^{-2}, & x > 0 \end{cases}$ [3]

(c) $f(x) = \begin{cases} x^2 \sin(1/x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$ [4]

Question 18

This question concerns the group

$$U = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}, ad \neq 0 \right\}$$

of 2×2 matrices, under matrix multiplication. (You are *not* asked to prove that U is a group.)

(a) Show that

$$H_1 = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}$$

is a subgroup of U. [3]

(b) Find the conjugate of the element $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ of H_1 by the element $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ of U.

Hence prove that H_1 is a normal subgroup of U. [3]

(c) You may now ASSUME that

$$H_2 = \left\{ \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} : b, d \in \mathbb{R}, d \neq 0 \right\}$$

is a subgroup of U.

Determine the left and right cosets of H_2 in U that contain the matrix $\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$. [4]

Question 19

(a) Determine the affine transformation t that maps the points $(0,0)$, $(1,0)$ and $(0,1)$ onto the points $(1,0)$, $(5,0)$ and $(1,9)$, respectively. [2]

(b) Determine the image of the point (x,y) under the affine transformation t^{-1}, the inverse of the transformation t that you found in part (a). [2]

(c) Determine the equation of the image of the ellipse

$$\frac{(x-1)^2}{16} + \frac{y^2}{81} = 1$$

under the affine transformation t^{-1} of part (b). [2]

(d) The tangents at two points P, Q of the ellipse in part (c) meet at M. Prove that the line joining M to the centre of the ellipse bisects the line segment PQ. [4]