Hence

$$\begin{aligned}\int \frac{dx}{x(\log_e x)^{4/3}} &= \int \frac{du}{u^{4/3}} \\ &= -3u^{-1/3} \\ &= -3(\log_e x)^{-1/3}.\end{aligned}$$ 1A

Let $f(x) = \dfrac{1}{x(\log_e x)^{4/3}}$, so that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log_e n)^{4/3}} = \sum_{n=2}^{\infty} f(n)$.

Now f is positive and decreasing on $[2, \infty[$, and $\frac{1}{2}$A

$$f(x) \to 0 \quad \text{as } x \to \infty.$$

Also,

$$\int_2^n f = \int_2^n \frac{dx}{x(\log_e x)^4/3}$$ $\frac{1}{2}$M for considering $\int_2^n f$

$$\begin{aligned} &= -3\left[\frac{1}{(\log_e x)^{1/3}}\right]_2^n \\ &= 3\left(\frac{1}{(\log_e 2)^{1/3}} - \frac{1}{(\log_e n)^{1/3}}\right) \\ &\le \frac{3}{(\log_e 2)^{1/3}}.\end{aligned}$$ 1A

Since the set

$$\left\{\int_2^n f : n \in \mathbb{N}\right\}$$

is bounded above, it follows from the Integral Test that the series converges. $\frac{1}{2}$M, $\frac{1}{2}$A

Question 14

(a) Here,

$$\begin{aligned} f(x) &= \frac{1}{x}, & f(2) &= \tfrac{1}{2}; \\ f'(x) &= -\frac{1}{x^2}, & f'(2) &= -\tfrac{1}{4}; \\ f''(x) &= \frac{2}{x^3}, & f''(2) &= \tfrac{1}{4}; \\ f'''(x) &= -\frac{6}{x^4}, & f'''(2) &= -\tfrac{3}{8}.\end{aligned}$$ 1A

Hence by Taylor's Theorem with $a = 2$ and $n = 3$, $\frac{1}{2}$M

$$\begin{aligned} T_3(x) &= \tfrac{1}{2} + \tfrac{-1/4}{1!}(x-2) + \tfrac{1/4}{2!}(x-2)^2 + \tfrac{-3/8}{3!}(x-2)^3 \\ &= \tfrac{1}{2} - \tfrac{1}{4}(x-2) + \tfrac{1}{8}(x-2)^2 - \tfrac{1}{16}(x-2)^3.\end{aligned}$$ $\frac{1}{2}$A

(b) Here,

$$f^{(4)}(x) = \frac{24}{x^5},$$

so that if $x \in \left[2, \frac{5}{2}\right]$, then

$$|f^{(4)}(x)| \le \tfrac{24}{2^5} = \tfrac{24}{32} = \tfrac{3}{4}.$$

It follows that we may take $M = \frac{3}{4}$ in the Remainder Estimate: 1A for value of M

$$|R_3(x)| \le \frac{\frac{3}{4}|x-2|^4}{4!}$$ 1M

$$\begin{aligned} &\le \frac{\frac{3}{4} \times \frac{1}{2^4}}{24} \\ &= \frac{3}{4 \times 16 \times 24} \\ &= \frac{1}{64 \times 8} \\ &= \tfrac{1}{512} < 10^{-2}.\end{aligned}$$ 1A