It follows that

$$\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} H_2 = \left\{ \begin{pmatrix} 2 & x \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R},\ y \neq 0 \right\}.$$ 1A

Similarly, the right coset is

$$H_2 \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} = \left\{ \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} : b, d \in \mathbb{R},\ d \neq 0 \right\}$$
$$= \left\{ \begin{pmatrix} 2 & 3+4b \\ 0 & 4d \end{pmatrix} : b, d \in \mathbb{R},\ d \neq 0 \right\}.$$ 1M

Again, $4d$ varies over $\mathbb{R}^*$, and $3+4b$ varies over $\mathbb{R}$, so that

$$H_2 \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} = \left\{ \begin{pmatrix} 2 & x \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R},\ y \neq 0 \right\}.$$ 1A

Question 19

(a) The required affine transformation t is of the form

$$t(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}.$$

First, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Next, the first column of the matrix $\mathbf{A}$ is

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix},$$

and its second column is

$$\begin{pmatrix} 1 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \end{pmatrix}.$$

Thus $\mathbf{A} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$.

It follows that the formula for t is

$$t(\mathbf{x}) = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ 2A

(b) Now the inverse of the matrix $\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$ is $\frac{1}{36} \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/9 \end{pmatrix}$, and

$$\begin{pmatrix} 1/4 & 0 \\ 0 & 1/9 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}.$$

It follows that the inverse affine transformation is

$$t^{-1}(\mathbf{x}) = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/9 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}.$$

Hence the image of the point (x, y) is $\left(\frac{1}{4}(x-1), \frac{1}{9}y\right)$. 1M, 1A

(c) Let $(x', y') = t^{-1}(x, y)$; then $x' = \frac{1}{4}(x-1)$ and $y' = \frac{1}{9}y$.

It follows that the image of the ellipse under the affine transformation t^{-1} has equation

$$(x')^2 + (y')^2 = 1;$$ 1M, 1A

that is, the unit circle.

(d) Since affine transformations preserve tangency, under the mapping t^{-1} the tangents to the ellipse at P, Q map to tangents to the unit circle at P', Q', say, that meet at M', the image of M under t. 2M