(c) We prove that f is continuous at 0, using the Squeeze Rule. Since

$$-1 \le \sin(1/x^2) \le 1, \qquad \text{for } x \ne 0,$$

we have

$$-x^2 \le x^2 \sin(1/x^2) \le x^2, \qquad \text{for } x \ne 0;$$ $\frac{1}{2}$A

and so, because $f(0) = 0$,

$$-x^2 \le f(x) \le x^2, \qquad \text{for } x \in \mathbb{R}.$$ $\frac{1}{2}$A

If we take

$$g(x) = -x^2 \quad \text{and} \quad h(x) = x^2,$$ $\frac{1}{2}$M

then

1. $g(x) \le f(x) \le h(x)$, for $x \in \mathbb{R}$; $\frac{1}{2}$A
2. $g(0) = h(0) = 0 = f(0)$; $\frac{1}{2}$A
3. g and h are continuous at 0, since they are polynomials. $\frac{1}{2}$A

Hence f is continuous at 0, by the Squeeze Rule. 1A ($\frac{1}{2}$ for quoting the correct rule)

Question 18

(a) SG1 Closure:

Let $\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} \in H$, so $a_1, a_2 \ne 0$.

$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1b_2 + b_1a_2 \\ 0 & a_1a_2 \end{pmatrix}.$$ 1A

Thus H_1 is closed under matrix multiplication as $a_1a_2 \ne 0$.

SG2 Identity:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ is in } H_1 \quad (\text{put } a = 1, b = 0).$$ $\frac{1}{2}$A

SG3 Inverses:

The inverse of $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is $\begin{pmatrix} \frac{1}{a} & -\frac{b}{a^2} \\ 0 & \frac{1}{a} \end{pmatrix}$, $\frac{1}{2}$A

and this is in H_1. $\frac{1}{2}$A

Thus H_1 is a subgroup of U. $\frac{1}{2}$M (for checking the correct axioms)

(b) The inverse of $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ is $\frac{1}{xz}\begin{pmatrix} z & -y \\ 0 & x \end{pmatrix} = \begin{pmatrix} \frac{1}{x} & -\frac{y}{xz} \\ 0 & \frac{1}{z} \end{pmatrix}$;

thus the required conjugate is

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}^{-1}$$ $\frac{1}{2}$M

$$= \begin{pmatrix} xa & xb + ya \\ 0 & za \end{pmatrix}\begin{pmatrix} \frac{1}{x} & -\frac{y}{xz} \\ 0 & \frac{1}{z} \end{pmatrix} = \begin{pmatrix} a & \frac{xb}{z} \\ 0 & a \end{pmatrix}.$$ 1A

Because $a \ne 0$, this final matrix belongs to H_1, since (for all values of a, b, x, z) it is of the correct form. Since the conjugate is in H_1, whatever element of U we use, it follows that H_1 is a normal subgroup of U. 1M, $\frac{1}{2}$A

(c) The left coset of H_2 in U that contains $\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ is the set

$$\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} H_2 = \left\{\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} : b, d \in \mathbb{R},\ d \ne 0\right\}$$

$$= \left\{\begin{pmatrix} 2 & 2b + 3d \\ 0 & 4d \end{pmatrix} : b, d \in \mathbb{R},\ d \ne 0\right\}.$$ 1M

As d runs through all numbers in $\mathbb{R}^*$, so also does $4d$; and then, for any given d, $2b + 3d$ runs through all real values since b does.