M203/R

The Open University

Second Level Course Examination 1999

Introduction to Pure Mathematics

Wednesday, 20 October, 1999 10.00 am–1.00 pm

Time allowed: 3 hours

In planning this paper, an allowance of 10 minutes was made for reading the questions.

There are **TWO** parts to this paper.

In Part I you should attempt as many questions as you can.
You should attempt no more than **THREE** questions in Part II.

70% of the available marks are assigned to Part I and 30% to Part II. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part I before starting Part II.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Attach your answer books together using the fastener provided.

The use of calculators is *not* permitted in this examination.

PART I

(i) *You should attempt as many questions as you can in this part.*

(ii) *Write your answers in the answer book provided, beginning each question on a new page.*

(iii) *Questions in this part do not necessarily carry equal marks. The mark allocation is indicated for each question.*

Question 1

Draw a sketch of the graph of the function f defined by

$$f(x) = \frac{4x+6}{3-x}.$$

Your sketch should include:

(a) any asymptotes to the graph;

(b) points where the graph crosses the axes. [4]

Question 2

Prove that the two sets A and B, defined below, are equal.

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 9\}$$
$$B = \{(3\cos t, 3\sin t) : t \in \mathbb{R}\}$$ [5]

Question 3

The position vectors of the points A, B and C are

$$\mathbf{a} = (-1, 3), \ \mathbf{b} = (5, 0) \text{ and } \mathbf{c} = (1, 2).$$

(a) Show that OC is perpendicular to AB.

(b) Write down the position vector of a general point on the line AB and hence show that C lies on AB.

(c) State the ratio in which C divides the line segment AB. [4]

Question 4

This question is about the system of linear equations

$$\begin{aligned} x + y - 2z &= 3, \\ 2x + y + z &= 4, \\ 3x + y + 4z &= 5. \end{aligned}$$

(a) Write down the augmented matrix for this system of linear equations.

(b) Find the row-reduced form of the matrix that you wrote down in part (a).

(c) Solve the equations by using the row-reduced form of the augmented matrix. [4]

Question 5

This question is about the matrix $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$.

(a) Find the eigenvalues of $\mathbf{A}$.

(b) Find the eigenvectors for $\mathbf{A}$.

(c) Show that the eigenvectors are orthogonal, and hence find an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors for $\mathbf{A}$.

(d) Write down an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D}$. [6]

Question 6

Determine the greatest lower bound of the set E, where

$$E = \left\{ 2 + \frac{3}{n^2} : n = 1, 2, 3, \ldots \right\}.$$ [4]

Question 7

Determine whether each of the following sequences $\{a_n\}$ is convergent, stating the limit of the sequence if it exists. You should name any result or test that you use.

(a) $a_n = \dfrac{2^n + n + n!}{3n^2 + 2(n!)}, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{3n^3 + n + 4^n}{4n^4 + 3^n}, \quad n = 1, 2, \ldots$ [6]

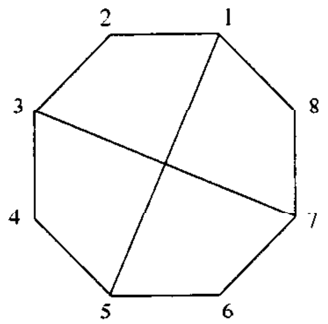
Question 8

Show that the following function is continuous on $\mathbb{R}$.

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} \cos 3x, & x < 0; \\ \dfrac{1}{x+1}, & x \geq 0 \end{cases}$$ [6]

Question 9

Let G be the symmetry group of the figure below, which is a regular octagon with two of its diagonals drawn.



(a) Write down the order of G.

(b) Using the numbering of the vertex locations shown on the figure, write down the elements of G in cycle form.

(c) List the conjugacy classes of G. [6]

Question 10

Let M be the set of all 2×2 matrices, which forms a group under addition of matrices. (You are NOT asked to prove this statement.)

The function

$$f : M \longrightarrow \mathbb{R}$$
$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \longmapsto a + d$$

maps each matrix in M to its trace.

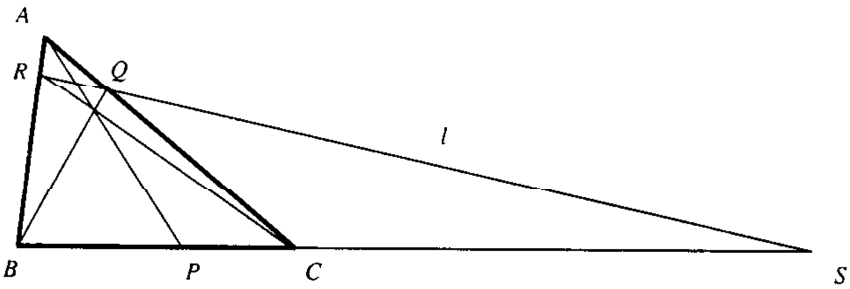
(a) Show that f is a group homomorphism.

(b) Find $\mathrm{Ker}(f)$.

(c) Justify that f is onto and hence explain why $M/\mathrm{Ker}(f) \cong \mathbb{R}$. [5]

Question 11

A line ℓ crosses the sides AB, BC, CA of a triangle ABC at the points R, S, Q, as shown in the diagram below. The ratio $AR : AB$ is $1 : 4$ and the ratio $AQ : AC$ is $1 : 3$.



The point P divides BC in the ratio $BP : PC = 3 : 2$.

(a) Decide whether the lines AP, BQ, and CR are concurrent.

(b) Determine the ratio in which S divides BC and hence show that the ratio $\dfrac{BP}{PC} \Big/ \dfrac{BS}{SC}$ is equal to -1.

You should state clearly any results that you use. [5]

Question 12

(a) Find a Möbius transformation $M : \mathbb{C} \to \mathbb{C}$ which sends $-2+3i$ to 0, $4+3i$ to 1, and $2+i$ to ∞.

(b) Using the Möbius transformation that you found in part (a), show that the points $-2+3i$, $4+3i$, $2+i$ and i lie on a circle. [5]

Question 13

Prove that the following limit exists and determine its value.

$$\lim_{x\to 0} \frac{3\sin 2x - x}{5e^{2x} - 5}$$ [5]

Question 14

The function f is defined on the interval $[0,4]$ by

$$f(x) = \begin{cases} 3, & x = 0, \\ 2x, & 0 < x < 4, \\ 1, & x = 4. \end{cases}$$

(a) Sketch the graph of f.

(b) Determine the values of the Riemann sums $L(f,P)$ and $U(f,P)$ for the partition P of $[0,4]$, where $P = \{[0,1],[1,3],[3,4]\}$. [5]

PART II

(i) *You should attempt* **no more than THREE** *questions from this part.*

(ii) *Each question carries 10 marks. The mark allocation for each section of a question is given in square brackets beside the section.*

(iii) *Start each question on a new page of your answer book.*

Question 15

The set of complex numbers

$$S = \left\{1, -1, \tfrac{1}{2}(1 + i\sqrt{3}), \tfrac{1}{2}(-1 + i\sqrt{3}), -\tfrac{1}{2}(1 + i\sqrt{3}), \tfrac{1}{2}(1 - i\sqrt{3})\right\}$$

forms a group under multiplication. (You are NOT asked to prove this statement.)

(a) Show that $H = \{1, -1\}$ is a subgroup of S and find the left cosets of H in S. [3]

(b) Show that S is cyclic. [4]

(c) Find all the subgroups of S. [2]

(d) Give an example of a symmetry group or modular arithmetic group from the course that is isomorphic to S. [1]

Question 16

This question is about the vector space P_3 of real polynomials of degree less than 3.

i.e. $P_3 = \left\{p(x) : p(x) = a + bx + cx^2; a, b, c \in \mathbb{R}\right\}$.

(a) Determine whether each of the following subsets of P_3 is a subspace of P_3.

(i) $S = \left\{p(x) : p(x) = a + cx^2; a, c \in \mathbb{R}\right\}$

(ii) $T = \{p(x) : p(0) = 3\}$. [4]

(b) Show that the function

$$\begin{aligned} t : P_3 &\longrightarrow P_3 \\ p(x) &\longmapsto p(x) - xp'(x) \end{aligned}$$

is a linear transformation, and determine the kernel and image of t. [6]

Question 17

Determine whether each of the following series is convergent. You should name any result or test that you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 2}{n^3 + n - 1}$ [3]

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n + 1)3^n}{n!}$ [3]

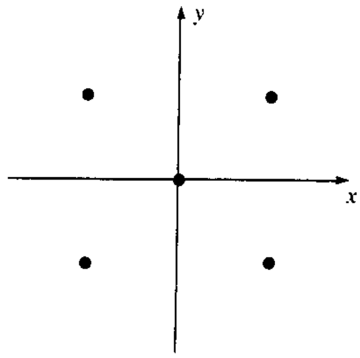
(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1 + \sin n}{1 + 2n^2}$ [4]

Question 18

The diagram below shows the set

$$S = \{(0,0), (1,1), (-1,1), (-1,-1), (1,-1)\}$$

of five points of $\mathbb{R}^2$.



(a) List the set G of rotations r_θ and reflections q_ϕ of the plane that map the set S to itself. [3]

You may assume, now, that G is a group and that it acts on the plane $\mathbb{R}^2$ in the natural way: that is, for $(x, y) \in \mathbb{R}^2$ and $g \in G$

$$g \wedge (x, y) = g(x, y).$$

(b) Write down the orbit and the stabiliser of

(i) $(0,0)$;

(ii) $(1,1)$;

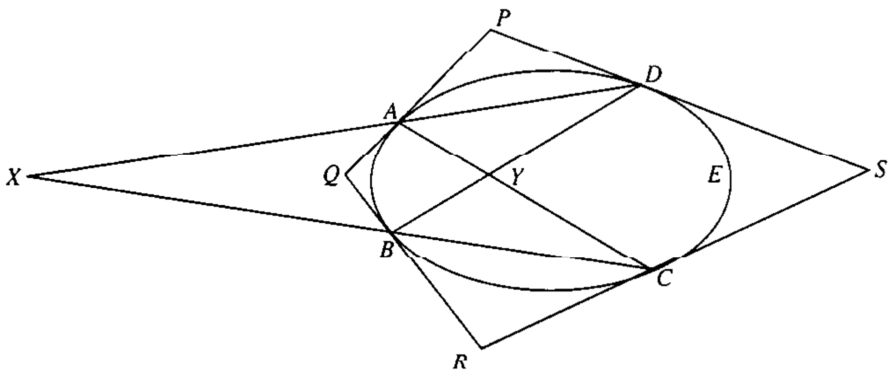
(iii) $(1,0)$;

(iv) $(2,1)$. [6]

(c) Explain why, under this action, an orbit cannot contain exactly three elements. [1]

Question 19

The projective conic E, in the following figure, has equation $x^2 + y^2 = z^2$, and it touches the quadrilateral $PQRS$ at the Points $A = [1, 0, 1]$, $B = [-1, 0, 1]$, $C = [0, -1, 1]$ and $D = [0, 1, 1]$. The Lines AD and BC meet at X, and the Lines AC and BD meet at Y.



(a) Find the equations of the tangents to E at A and B, and hence determine the Point Q where they meet. [2]

(b) Find the equations of the tangents to E at C and D, and hence determine the Point S where they meet. [2]

(c) Write down the equations of the Lines AD and BC, and hence determine the Point X where they meet. [2]

(d) Write down the equations of the Lines AC and BD, and hence determine the Point Y where they meet. [2]

(e) Show that S, Q, X, Y are collinear and calculate the cross ratio $(S\,Q\,X\,Y)$. [2]

Question 20

This question concerns the linear flow for which the velocity function is

$$V(x, y) = (-13x + 16y, -8x + 11y).$$

(a) Write down

(i) the matrix $\mathbf{A}$ of the flow;

(ii) first order differential equations satisfied by the co-ordinate functions f and g of any flow function $\alpha = (f, g)$ for this flow;

(iii) a second order differential equation satisfied by both f and g. [3]

(b) Find the general solution of the differential equation in part (a) (iii). [2]

(c) Find the general form of the flow function α for V. [3]

(d) Determine the flow function α for V that satisfies $\alpha(0) = (1, 0)$. [2]

[END OF QUESTION PAPER]