M203/L

The Open University

Second Level Course Examination 1998
Introduction to Pure Mathematics

Monday, 19 October, 1998 10.00 am – 1.00 pm

Time allowed: 3 hours

In planning this paper, an allowance of 10 minutes was made for reading the questions.

There are **TWO** parts to this paper.

In Part I you should attempt as many questions as you can.
You should attempt no more than **THREE** questions in Part II.

70% of the available marks are assigned to Part I and 30% to Part II. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part I before starting Part II.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Attach your answer books together using the fastener provided.

The use of calculators is *not* permitted in this examination.

PART I

(i) *You should attempt as many questions as you can in this part.*

(ii) *Write your answers in the answer book provided, beginning each question on a new page.*

(iii) *Questions in this part do not necessarily carry equal marks. The mark allocation is indicated for each question.*

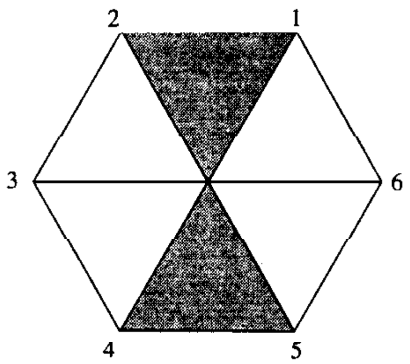
Question 1

Draw a sketch of the graph of the function f defined by

$$f(x) = \begin{cases} e^x, & -\infty < x < 0; \\ \cos x, & 0 \le x \le \frac{\pi}{2}; \\ 1 + x, & \frac{\pi}{2} < x < \infty. \end{cases}$$ [4]

Question 2

This question concerns the symmetry group $S(F)$ of the figure F below, which is a modified regular hexagon.



(a) Write down the number of symmetries of F.

(b) Using the numbering of the vertex locations shown on the figure, write down the elements of the symmetry group $S(F)$.

(c) Is $S(F)$ cyclic? Give a brief reason for your answer. [5]

Question 3

The complex numbers z_1 and z_2 are defined as $z_1 = 5 - 12i$ and $z_2 = 2 - 7i$.

(a) Sketch the positions of z_1 and z_2 on the Argand plane.

(b) Calculate the modulus of z_1.

(c) Evaluate the complex number $\frac{z_2}{z_1}$ giving your answer in the form $a + ib$. [4]

Question 4.

The set S is defined by $S = \{(a, b, 2a - b) : a, b \in \mathbb{R}\}$.

(a) Show that S is a subspace of $\mathbb{R}^3$.

(b) Show that the vectors $(1, 0, 2)$ and $(1, -1, 3)$ belong to S. Express the general vector $(a, b, 2a - b)$ as a linear combination of these two vectors. Explain why the set $\{(1, 0, 2), (1, -1, 3)\}$ is a basis for S. [5]

Question 5

Find the matrix of the linear transformation

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto (2x + 3y, x - 4y)$$

with respect to

(a) the standard basis in both the domain and the codomain;

(b) the basis $\{(3, 1), (1, -1)\}$ in the domain and the standard basis in the codomain;

(c) the basis $\{(3, 1), (1, -1)\}$ in both domain and codomain. [5]

Question 6

Find the solution set of the inequality

$$\frac{1}{x} < \frac{2}{x-1}.$$ [4]

Question 7

Determine whether each of the sequences $\{a_n\}$ below is convergent, stating the limit of the sequence if it exists. You should name any result or test you use.

(a) $a_n = \dfrac{n^3 + 3(n!) + 1}{n^2 + 2(n!)} \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{3^n + n^2}{2^n + n^3} \quad n = 1, 2, \ldots$ [6]

Question 8

Determine whether each of the series below is convergent. You should name any result or test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2 + 1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n + 1}{(2n)!}$ [6]

Question 9

The set $G = \{1, 5, 7, 11, 13, 17, 19, 23\}$ forms a group under multiplication modulo 24. (You are NOT asked to prove this statement.)

(a) Show that $H = \{1, 5\}$ is a subgroup of G.

(b) Write down the left cosets of H in G.

(c) Explain why H is a normal subgroup of G.

(d) Write down a group from the course isomorphic to the quotient group G/H. Give a brief reason for your answer. [6]

Question 10

The group $S(\triangle) = \{e, a, b, r, s, t\}$ is the symmetry group of an equilateral triangle. The function

$$f : (S(\triangle), \circ) \longrightarrow (\mathbb{Z}_8, +_8)$$

is a homomorphism and $f(s) = 4$.

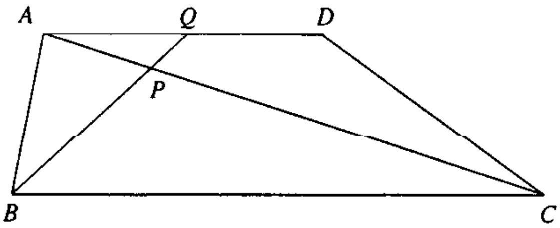
(a) Write down $f(e)$.

(b) Write down the image of each element of $S(\triangle)$ under f.

(c) Write down $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$. [5]

Question 11

The figure shows a trapezium $ABCD$ in which the length of BC is twice the length of AD. The line from B to the midpoint Q of AD crosses the diagonal AC at P.



Let $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the affine transformation which sends A to $(0, 1)$, B to $(0, 0)$ and C to $(1, 0)$.

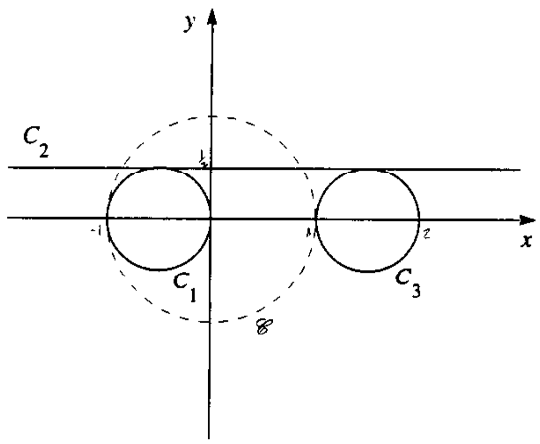
(a) Sketch the image of the figure above under t.

(b) Calculate the coordinates of the image of P under t.

(c) Hence determine the ratio $BP : PQ$. [5]

Question 12

Let C_1 and C_3 be circles of radius $\frac{1}{2}$ centered at $(-\frac{1}{2}, 0)$ and $(\frac{3}{2}, 0)$, respectively, and let C_2 be the extended line with equation $y = \frac{1}{2}$.



Sketch, on one diagram, the images of C_1, C_2 and C_3 under inversion in the unit circle $\mathscr{C}$. Mark clearly which image is which. [5]

Question 13

Calculate the Taylor polynomial $T_2(x)$ for the function $f(x) = x/(x-1)$ at 2.

Show that $T_2(x)$ approximates $f(x)$ to within $\frac{1}{8}$ on the interval $[2, 2\frac{1}{2}]$. [5]

Question 14

(a) Determine the barrier lines (if any) for the flow with velocity function

$$V(x, y) = (x + 5y, x - 3y),$$

and the direction of the flow on each barrier line.

(b) Hence draw a rough sketch of the flow. [5]

PART II

(i) *You should attempt* **no more than THREE** *questions from this part.*

(ii) *Each question carries 10 marks. The mark allocation for each section of a question is given in square brackets beside the section.*

(iii) *Start each question on a new page of your answer book.*

Question 15

Let A, B and C be the sets defined by

$$A = \{(x, y) \in \mathbb{R}^2 : y = x^2 + 4x\};$$
$$B = \{(t-2, t^2-4) : t \in \mathbb{R}\};$$
$$C = \{(t^2, t^4 + 4t^2) : t \in \mathbb{R}\}.$$

(a) Draw a sketch of the set A. [2]

(b) Show that $A = B$. [6]

(c) Show $A \neq C$. [2]

Question 16

The matrix $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$ represents a linear transformation f with respect to the standard basis in both the domain and the codomain.

(a) Find the row-reduced form of the matrix $\mathbf{A}$. [3]

(b) Hence determine the kernel of f. [2]

(c) Determine the dimension of $\text{Im}(f)$, and find a basis for $\text{Im}(f)$. [2]

(d) Determine whether the vector $\mathbf{v} = (1, 7, 2)$ belongs to $\text{Im}(f)$. [3]

Question 17

(a) Determine the set of points at which the following function is continuous.

$$f(x) = \begin{cases} 3\sin x, & x < 0; \\ e^x - x^2 - 1, & x \geq 0. \end{cases}$$ [6]

(b) Show that the equation $1 + \sin x = \frac{x}{2}$ has exactly one solution in $[\frac{\pi}{2}, \pi]$. State clearly any results you use. [4]

Question 18

This question concerns the set of matrices

$$U = \left\{ \begin{pmatrix} a & 2(b-a) \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R}^* \right\}.$$

(a) Show that U forms a group under matrix multiplication. [3]

The group U acts on the plane $\mathbb{R}^2$ as follows:

$$\begin{pmatrix} a & 2(b-a) \\ 0 & b \end{pmatrix} \wedge (x, y) = (ax + 2(b-a)y, by)$$

for each $(x, y) \in \mathbb{R}^2$. (You are NOT asked to verify that this is a group action. Essentially the action is matrix multiplication but you are expected to work with the group action on the plane.)

(b) Find the orbit of

(i) $(1, 0)$;

(ii) $(0, 1)$.

Give a geometric description of each of these orbits and deduce the set of orbits of the action. [4]

(c) Find the stabiliser of

(i) $(1, 0)$;

(ii) $(0, 1)$. [3]

Question 19

Let E be the non-degenerate projective conic with equation

$$y^2 = xz.$$

(a) Show that E passes through the Points $A = [1, 0, 0]$, $B = [0, 0, 1]$ and $C = [1, 1, 1]$. [1]

(b) Use Joachimsthal's notation to show that the tangent at an arbitrary Point $[a, b, c]$ on E has equation

$$cx - 2by + az = 0.$$

Hence show that the tangents to E at the Points A and B meet at the Point $P = [0, 1, 0]$. [3]

(c) Show that:

(i) the Line PC meets the Line AB at the Point $Q = [1, 0, 1]$;

(ii) the Line PC meets E (again) at the Point $D = [1, -1, 1]$. [4]

(d) Calculate the cross-ratio $(CDPQ)$. [2]

Question 20

(a) (i) Sketch the graph of the function

$$f(x) = \begin{cases} x^2, & x \leq 1; \\ e^{x-1}, & x > 1. \end{cases}$$ [1]

(ii) Prove (carefully) that $f'_L(1)$ and $f'_R(1)$ exist, and determine their values. [4]

(iii) Determine whether $f'(1)$ exists, and give a brief justification of your answer. [1]

(b) Prove that the limit

$$\lim_{x \to 3} \frac{(3x-5)^{1/2} - (11-x)^{1/3}}{x-3}$$

exists, and determine its value. [4]

[END OF QUESTION PAPER]