M203/G

The Open University

Second Level Course Examination 1997
Introduction to Pure Mathematics

Thursday, 16 October, 1997 10.00 am – 1.00 pm

Time allowed: 3 hours

In planning this paper, an allowance of 10 minutes was made for reading the questions.

There are **TWO** parts to this paper.

In Part I you should attempt as many questions as you can.
You should attempt no more than **THREE** questions in Part II.

70% of the available marks are assigned to Part I and 30% to Part II. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part I before starting Part II.

At the end of the examination

Check that you have written your name, personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Attach your answer books together using the fastener provided.

The use of calculators is *not* permitted in this examination.

PART I

(i) *You should attempt as many questions as you can in this part.*

(ii) *Write your answers in the answer book provided, beginning each question on a new page.*

(iii) *Questions in this part do not necessarily carry equal marks. The mark allocation is indicated for each question.*

Question 1

Draw a sketch of the function f defined by

$$f(x) = \frac{3-x}{3-2x}.$$

Your sketch should include:

(a) any asymptotes for the graph;

(b) any points where the graph crosses the axes. [4]

Question 2

The position vectors of points A and B are $\mathbf{a} = (4, -3)$ and $\mathbf{b} = (1, 3)$, respectively.

(a) Draw a sketch showing the points A and B in the plane, and the line ℓ through A and B.

(b) Find the position vector $\mathbf{r}$ of a general point on the line ℓ.

(c) Find the point P on ℓ whose position vector is perpendicular to ℓ. Mark P on your sketch. [5]

Question 3

(a) Determine the row-reduced form of the matrix

$$\begin{pmatrix} 1 & 3 & -2 \\ 2 & 1 & 1 \\ 1 & -7 & 8 \end{pmatrix}.$$

(b) Hence or otherwise find the solution set of the equations

$$\begin{aligned} x + 3y - 2z &= 0 \\ 2x + y + z &= 0 \\ x - 7y + 8z &= 0. \end{aligned}$$ [5]

Question 4

Find the matrix of the linear transformation

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x - 2y, 3x + y) \end{aligned}$$

with respect to

(a) the standard basis in both domain and codomain;

(b) the basis $\{(1, -2), (2, 1)\}$ in the domain and the standard basis in the codomain;

(c) the basis $\{(1, -2), (2, 1)\}$ in both domain and codomain. [5]

Question 5

Determine the least upper bound of the set E where

$$E = \left\{5 - \frac{4}{n^2} : n = 1, 2, 3, \ldots\right\}.$$ [4]

Question 6

Determine whether each of the following series is convergent. (You should name any result or test that you use.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{n+3n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!}$ [6]

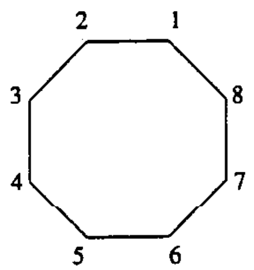
Question 7

Show that the following function is continuous on the whole of $\mathbb{R}$.

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} 2e^x - 1, & x < 0, \\ \cos 2x, & x \geq 0. \end{cases}$$ [6]

Question 8

The figure F below is a regular octagon.



(a) Write down, in cycle notation, using the numbering of the locations of the vertices shown, the symmetry g of F that is a rotation through an angle $3\pi/4$ anticlockwise about the centre of the figure.

(b) Write down, in cycle notation, the symmetry h of F that is a reflection in the axis through locations 3 and 7.

(c) Find the conjugate ghg^{-1}, and identify this conjugate as a symmetry of the octagon.

(d) Are (14)(23)(58)(67) and (15)(26)(37)(48) conjugate as symmetries within $S(F)$? Justify your answer briefly. [5]

Question 9

The set $G = \{1, 4, 7, 10, 13, 16, 19, 22, 25\}$ forms a group under multiplication modulo 27. You are NOT asked to prove this result.

(a) Show that G is a cyclic group.

(b) State whether it is possible to find a homomorphism from $(G, \times_{27})$ onto (i) $(\mathbb{Z}_3, +_3)$; (ii) $(\mathbb{Z}_4, +_4)$; (iii) $(\mathbb{Z}_6, +_6)$, justifying your answers briefly.

(c) Write down the kernel of each homomorphism from part (b). [5]

Question 10

The group $G = \{r_0, r_{\pi/2}, r_\pi, r_{3\pi/2}, q_0, q_{\pi/4}, q_{\pi/2}, q_{3\pi/4}\}$, isomorphic to $S(\square)$, acts on the plane in the natural way:

$$g \wedge (x, y) = g(x, y).$$

That is, $g \wedge (x, y)$ is the image of (x, y) under the transformation g. (You are NOT asked to prove any of these statements.)

(a) Find the orbit of $(1, 0)$ and the orbit of $(1, -1)$.

(b) Find the stabilizer of $(1, 0)$ and the stabilizer of $(1, -1)$. [5]

Question 11

(a) Find a Möbius transformation $M : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ which sends 4 to 0, $4 + 4i$ to 1, and $-2 + 4i$ to ∞.

(b) Using the Möbius transformation that you found in part (a), determine whether the points 4, $4 + 4i$, $-2 + 4i$ and $-1 - i$ lie on a circle. [5]

Question 12

Find a matrix **A** associated with the projective transformation which maps the Points

$$[1, 0, 0], [0, 1, 0], [0, 0, 1] \text{ and } [1, 1, 1],$$

to the Points

$$[1, 1, -1], [1, 2, -1], [1, 0, -3] \text{ and } [2, -1, -6],$$

respectively. [5]

Question 13

Determine the Taylor polynomial $T_2(x)$ for the function

$$f(x) = (5 - x)^{3/2}$$

at 1. Show that $T_2(x)$ approximates $f(x)$ to within $\frac{1}{48}$ on the interval $[1, 2]$. [5]

Question 14

This question concerns the integral

$$I_n = \int_1^e x^4 (\log_e x)^n \, dx.$$

(a) Evaluate the integral I_0.

(b) Prove that, for $n \geq 1$,

$$I_n = \frac{e^5}{5} - \frac{n}{5} I_{n-1}.$$

(c) Find the value of I_2. [5]

PART II

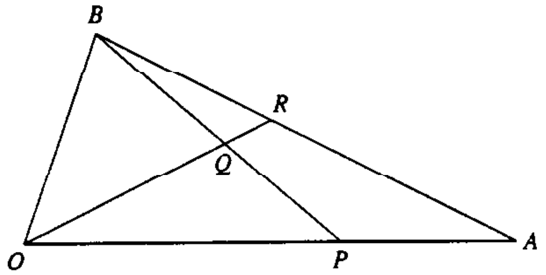
(i) *You should attempt* **no more than THREE** *questions from this part.*

(ii) *Each question carries 10 marks. The mark allocation for each section of a question is given in square brackets beside the section.*

(iii) *Start each question on a new page of your answer book.*

Question 15

The triangle OAB has vertices at O (the origin) and at points A and B with position vectors **a** and **b** respectively. The point P divides OA in the ratio 3 : 2 and Q is the mid-point of BP. The line OQ meets AB at R.



(a) Write down the position vectors of the points P and Q in terms of **a** and **b**. [2]

(b) Find the vector form of the equations of the lines AB and OQ, and hence find the position vector of the point R. In what ratio does R divide AB? [5]

(c) If $\mathbf{a} = (11, -7)$ and $\mathbf{b} = (-5, 1)$, find the cosine of the angle $B\hat{O}R$. [3]

Question 16

This question concerns the set

$$S = \{(a, b, 2a + b, a - 3b) : a, b \in \mathbb{R}\}.$$

(a) Prove that S is a subspace of $\mathbb{R}^4$. [3]

(b) (i) Show that the vectors $(1, 1, 3, -2)$ and $(1, -1, 1, 4)$ belong to S.

(ii) Express the vector $(a, b, 2a + b, a - 3b)$ as a linear combination of the two vectors $(1, 1, 3, -2)$ and $(1, -1, 1, 4)$.

(iii) Explain why the set $\{(1, 1, 3, -2), (1, -1, 1, 4)\}$ is a basis for S. [3]

(c) Find an orthogonal basis for S that includes the vector $(1, 1, 3, -2)$. [2]

(d) Express the vector $(5, 2, 12, -1)$ as a linear combination of the vectors of the orthogonal basis that you found in part (c). [2]

Question 17

Determine whether each of the following sequences $\{a_n\}$ is convergent, stating the limit of the sequence (if a limit exists). You should state any result or test that you use.

(a) $a_n = \dfrac{3^n + n!}{2(n!) + 2^n}, \quad n = 1, 2, \ldots$ [3]

(b) $a_n = \dfrac{3^n + n + 1}{2n + 2^n - 5}, \quad n = 1, 2, \ldots$ [4]

(c) $a_n = \dfrac{(-1)^n n^2}{2n^2 + n + 1}, \quad n = 1, 2, \ldots$ [3]

Question 18

The group $G = \{e, a, b, c, d, f, g, h, i, j, k, l, m, n, o, p\}$ is defined by the following group table. (You are NOT expected to show that G is a group.)

	e	a	b	c	d	f	g	h	i	j	k	l	m	n	o	p
e	e	a	b	c	d	f	g	h	i	j	k	l	m	n	o	p
a	a	b	c	d	f	g	h	e	p	i	j	k	l	m	n	o
b	b	c	d	f	g	h	e	a	o	p	i	j	k	l	m	n
c	c	d	f	g	h	e	a	b	n	o	p	i	j	k	l	m
d	d	f	g	h	e	a	b	c	m	n	o	p	i	j	k	l
f	f	g	h	e	a	b	c	d	l	m	n	o	p	i	j	k
g	g	h	e	a	b	c	d	f	k	l	m	n	o	p	i	j
h	h	e	a	b	c	d	f	g	j	k	l	m	n	o	p	i
i	i	j	k	l	m	n	o	p	e	a	b	c	d	f	g	h
j	j	k	l	m	n	o	p	i	h	e	a	b	c	d	f	g
k	k	l	m	n	o	p	i	j	g	h	e	a	b	c	d	f
l	l	m	n	o	p	i	j	k	f	g	h	e	a	b	c	d
m	m	n	o	p	i	j	k	l	d	f	g	h	e	a	b	c
n	n	o	p	i	j	k	l	m	c	d	f	g	h	e	a	b
o	o	p	i	j	k	l	m	n	b	c	d	f	g	h	e	a
p	p	i	j	k	l	m	n	o	a	b	c	d	f	g	h	e

(a) Find H, the cyclic subgroup generated by the element b. [1]

(b) Show that $K = \{e, d, i, m\}$ is a subgroup. [3]

(c) One of the subgroups H and K is a normal subgroup of G and one is not a normal subgroup of G. Identify which is a normal subgroup and which is not, giving a reason for each decision. [3]

(d) For the normal subgroup, write down the elements of the quotient group formed from G by this subgroup. [1]

(e) Write down a symmetry group or a modular arithmetic group that is isomorphic to the quotient group that you identified in part (d) justifying your answer briefly. [2]

Question 19

Let ABC be the triangle with vertices at $A(0,1)$, $B(0,0)$, $C(1,0)$, and let P, Q, R be the mid-points of the sides BC, CA, BA, respectively.

(a) Find the equations of the lines AP and CR, and hence determine the point X where AP and CR meet. [3]

(b) Show that X lies on the line BQ. [1]

(c) Calculate the ratios $\dfrac{AX}{XP}$, $\dfrac{BX}{XQ}$ and $\dfrac{CX}{XR}$. [2]

(d) Use your answers to the previous parts of the question to explain why the medians of any triangle DEF meet at a point Y which lies two-thirds of the way along the line segments from each of the vertices D, E, F to the mid-points S, T, U of the opposite sides. [4]

Question 20

This question concerns the linear flow whose velocity function is

$$V(x, y) = (2y - 7x, x - 8y).$$

(a) Write down:

(i) the matrix $\mathbf{A}$ of the flow;

(ii) first order differential equations satisfied by the co-ordinate functions f and g of any flow function $\alpha = (f, g)$ for this flow;

(iii) a second order differential equation satisfied by both f and g. [2]

(b) Find any barrier lines for the flow. [3]

(c) Find the general solution of the differential equation in part (a) (iii). [1]

(d) Determine the general form of the flow function α for V. [2]

(e) Determine the particular flow function for V that satisfies $\alpha(0) = (1, 1)$. [2]

[END OF QUESTION PAPER]