1996

PART I

(i) *You should attempt as many questions as you can in this part.*

(ii) *Write your answers in the answer book provided, beginning each question on a new page.*

(iii) *Questions in this part do not necessarily carry equal marks. The mark allocation is indicated for each question.*

Question 1

Draw a sketch of the graph of the function f defined by

$$f(x) = \frac{3-2x}{x+1}.$$

Your sketch should include:

(a) any asymptotes for the graph;

(b) any points where the graph crosses the axes. [4]

Question 2

Express the complex number $z = \left(\frac{7-i}{3+i}\right)^2$ in Cartesian form.

Write down $|z|$ and $\sin\theta$ where θ is the argument of z. [5]

Question 3

Find the matrix of the linear transformation

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (2x + y, 3x - y) \end{aligned}$$

with respect to

(a) the standard basis in both domain and codomain;

(b) the basis $\{(1, 1), (2, -3)\}$ in the domain and the standard basis in the codomain;

(c) the basis $\{(1, 1), (2, -3)\}$ in both domain and codomain. [5]

Question 4

The set S is defined by $S = \{(a, b, 3b - a) : a, b \in \mathbb{R}\}$.

(a) Show that S is a subspace of $\mathbb{R}^3$.

(b) Show that the vectors $(1, 0, -1)$ and $(1, 1, 2)$ belong to S. Express the general vector $(a, b, 3b - a)$ as a linear combination of these two vectors.

Explain why the set $\{(1, 0, -1), (1, 1, 2)\}$ is a basis for S. [5]

Question 5

Find the solution set of the inequality

$$\frac{3x}{x^2 - 4} < 1.$$ [5]

Question 6

Determine whether or not each of the following sequences $\{a_n\}$ converges, naming any result or test that you use. If it does converge, find its limit.

(a) $a_n = \dfrac{n^3 - 11n + 4(n!)}{2n^2 + n! - 6}$

(b) $a_n = \dfrac{2n^4 + 5^n + 8}{n^3 + 11n + 3^n}$ [6]

Question 7

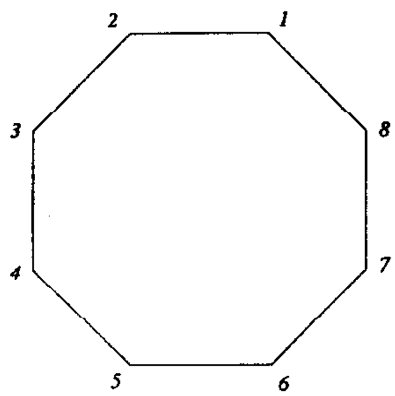
Show that the function

$$f : x \longmapsto \begin{cases} \sin 2\pi x, & x \leq 1, \\ x^2 - 1, & x > 1, \end{cases}$$

is continuous on $\mathbb{R}$. You should name any test or result that you use to justify your answer. [5]

Question 8

This question concerns the symmetry group G of the regular octagon, shown below.



Let $g \in G$ be the anticlockwise rotation of the octagon through an angle of $\frac{3\pi}{4}$ about its centre, and let $h \in G$ be the reflection of the octagon in the line through the vertices at locations 3 and 7.

(a) Write g, g^2 and h in cycle form, using the numbering of the locations of the vertices as shown above. State the order of g.

(b) Express the conjugate ghg^{-1}, of h by g, in cycle form, and describe ghg^{-1} geometrically.

(c) Are the symmetries $k = (15)(26)(37)(48)$ and $l = (12)(38)(47)(56)$ conjugate? Give a brief reason for your answer. [5]

Question 9

In this question

$$G = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$$

which is a group under the operation of multiplication modulo 21. (You are NOT asked to prove this result.)

(a) Find a subgroup of G of order 6.

(b) Write down the homomorphism property as it would apply to the specific case of a function

$$f : G \longrightarrow \mathbb{Z}_4.$$

(c) Specify (by giving the image of each element of G) a homomorphism

$$f : G \longrightarrow \mathbb{Z}_4$$

that has as kernel the subgroup that you found in part (a). [5]

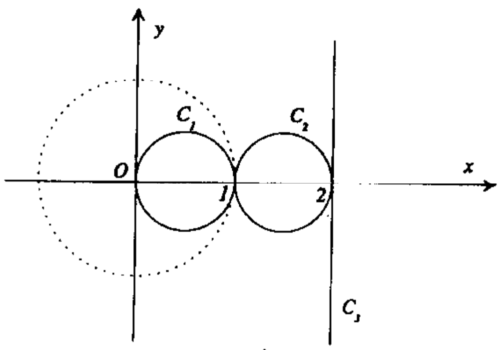
Question 10

(a) Find an affine transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which sends the points $(0,0)$, $(1,0)$, $(0,1)$ to the points $(2,3)$, $(5,7)$, $(3,5)$, respectively.

(b) Find the inverse of t, and hence or otherwise find the image of the line $y = 2x$ under t. [6]

Question 11

Let C_1 and C_2 be circles of radius $\frac{1}{2}$ centred at $\left(\frac{1}{2}, 0\right)$ and $\left(\frac{3}{2}, 0\right)$, respectively, and let C_3 be the extended line with equation $x = 2$.



Draw a sketch showing the images of C_1, C_2 and C_3 under inversion in the unit circle $\mathscr{C}$. Mark clearly which image is which. [4]

Question 12

(a) Find the equation of the Line which passes through the Points $A = [1, 2, 3]$ and $B = [1, 1, 1]$. Check that the Points $C = [2, -1, -4]$ and $D = [3, 1, -1]$ lie on this Line.

(b) Calculate the cross-ratio $(ABCD)$. [5]

Question 13

Prove that

$$\lim_{x \longrightarrow 1} \frac{x-(3-2x)^{1/2}}{x^2+x-2}$$

exists and determine its value. [5]

Question 14

This question concerns $I_n = \int_1^e x\,(\log_e x)^n \, dx$, $n \geq 0$.

(a) Evaluate I_0.

(b) Show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$, where $n \geq 1$, and hence evaluate I_2. [5]

PART II

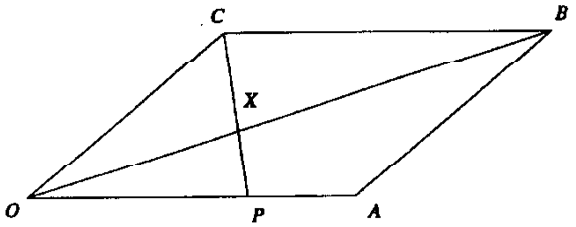
(i) *You should attempt* **no more than THREE** *questions from this part.*

(ii) *Each question carries 10 marks. The mark allocation for each section of a question is given in square brackets beside the section.*

(iii) *Start each question on a new page of your answer book.*

Question 15

The parallelogram $OABC$ has vertices at O (the origin) and at points A, B, C with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. The point P lies two-thirds of the way along OA from O. The lines OB, CP intersect at X.



(a) Write down the position vectors $\mathbf{b}$ and $\mathbf{p}$ of the points B and P, respectively, in terms of $\mathbf{a}$ and $\mathbf{c}$. [2]

(b) Find the vector form of the equations of the lines OB and CP, in terms of $\mathbf{a}$ and $\mathbf{c}$ and hence find the position vector of the point X. In what ratio does X divide OB? [4]

(c) If $\mathbf{a} = (3,1)$ and $\mathbf{c} = (1,2)$ show that CP is perpendicular to OB, and find the cosine of the angle between OC and OB. [4]

Question 16

The matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ represents a linear transformation f with respect to the standard basis in both the domain and the codomain.

(a) Determine the characteristic equation of $\mathbf{A}$ and solve this equation to find the eigenvalues of f. [3]

(b) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of f. [4]

(c) The matrix $\mathbf{C}$ of f with respect to the eigenvector basis that you chose in part (b) in both domain and codomain can be expressed as $\mathbf{C} = \mathbf{P}^T\mathbf{A}\mathbf{P}$. Find a suitable matrix $\mathbf{P}$, and write down the matrix $\mathbf{C}$. [3]

Question 17

Determine whether or not each of the following series is convergent, naming any result or test that you use.

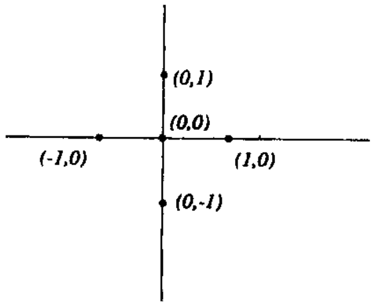
(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2+n-1}$ [3]

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 2^n}{n!}$ [3]

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2\cos n}{2n^2+1}$ [4]

Question 18

The diagram below shows the five points $(0,0)$, $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$ in $\mathbb{R}^2$.



(a) List the set G of rotations, r_θ, and reflections, q_ϕ, of the plane that fix the set $\{(0,0),(1,0),(0,1),(-1,0),(0,-1)\}$ (as a set). [3]

You may assume now that the group G acts on the plane $\mathbb{R}^2$ in the natural way; that is, for $g \in G$ and $x \in \mathbb{R}^2$,

$$g \wedge x = g(x).$$

(You are NOT asked to prove that this is a group action.)

(b) Find the orbit and the stabiliser of

(i) $(0,0)$,

(ii) $(0,1)$,

(iii) $(1,1)$,

(iv) $(2,1)$. [7]

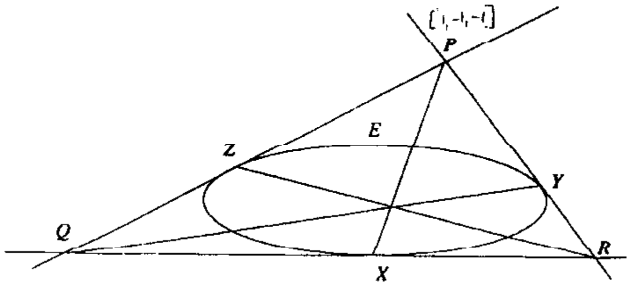
Question 19

(a) Let E be the non-degenerate projective conic with equation

$$xy + yz + zx = 0.$$

(i) Show that E passes through the Points $X = [1,0,0]$, $Y = [0,1,0]$ and $Z = [0,0,1]$.

(ii) Find an equation for the tangent to E at each of the Points X, Y and Z.



(iii) Determine the Point P where the tangents at Y and at Z meet. Hence or otherwise find the Point Q where the tangents at X and at Z meet and the Point R where the tangents at X and at Y meet.

(iv) Show that the Lines XP, YQ and ZR meet at a single Point. [8]

(b) Let ABC be an arbitrary triangle which touches a non-degenerate projective conic F at Points U, V, W. Use your answer to part (a) to explain why the Lines UA, VB, WC are concurrent. [2]

Question 20

This question concerns the linear flow for which the velocity function is

$$V(x, y) = (3x + 2y, x + 4y).$$

(a) Write down

(i) the matrix **A** of the flow;

(ii) the first order differential equations satisfied by the co-ordinate functions f and g of any flow function $\alpha = (f, g)$ for this flow;

(iii) a second order differential equation satisfied by both f and g. [3]

(b) Find the general solution of the differential equation in part (a)(iii). [3]

(c) Determine the flow function α for V which satisfies $\alpha(0) = (1, 4)$. [4]

[END OF QUESTION PAPER]