M203/H

The Open University

Second Level Course Examination 2000

Introduction to Pure Mathematics

Thursday, 12 October, 2000 2.30 pm–5.30 pm

Time allowed: 3 hours

In planning this paper, an allowance of 10 minutes was made for reading the questions.

There are **TWO** parts to this paper.

In Part I you should attempt as many questions as you can.
You should attempt no more than **THREE** questions in Part II.

70% of the available marks are assigned to Part I and 30% to Part II. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part I before starting Part II.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Attach your answer books together using the fastener provided.

The use of calculators is *not* permitted in this examination.

PART I

(i) *You should attempt as many questions as you can in this part.*

(ii) *Write your answers in the answer book provided, beginning each question on a new page.*

(iii) *Questions in this part do not necessarily carry equal marks. The mark allocation is indicated for each question.*

Question 1

Draw a sketch of the graph of the function f defined by

$$f(x) = \frac{3x+2}{2x-1}.$$

Your sketch should include:

(a) any asymptotes to the graph;

(b) any points where the graph crosses the axes. [4]

Question 2

Let $z = \dfrac{6-2i}{2+i}$.

(a) Express z in Cartesian form.

(b) Find the modulus and argument of z. [4]

Question 3

The position vectors of points A and B are $\mathbf{a} = (2,1)$ and $\mathbf{b} = (4,-1)$, respectively.

(a) Draw a sketch showing the points A and B in the plane, and the line l through A and B.

(b) Find the position vector $\mathbf{r}$ of a general point on the line l.

(c) Find the point P on l whose position vector is perpendicular to l. [5]

Question 4

This question concerns the system of linear equations

$$\begin{cases} x + 2y + z = 3, \\ -x + 4y + z = -3, \\ x + 5y + 3z = 6. \end{cases}$$

(a) Write down the augmented matrix for this system of linear equations.

(b) Find the row-reduced form of the matrix that you wrote down in part (a).

(c) Solve the system of equations by using the row-reduced form of the augmented matrix. [4]

Question 5

Find the matrix of the linear transformation

$$t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto (x + 2y, 3x - y)$$

with respect to

(a) the standard basis for both the domain and the codomain;

(b) the basis $\{(1, 1),\ (2, 1)\}$ for the domain and the standard basis for the codomain;

(c) the basis $\{(1, 1),\ (2, 1)\}$ for both the domain and the codomain. [5]

Question 6

Find the solution set of the inequality

$$\frac{3}{x-2} < \frac{x-2}{3}.$$ [5]

Question 7

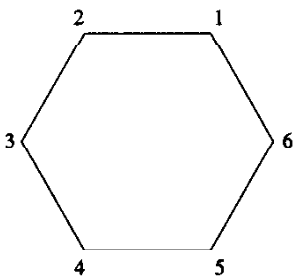
Determine whether each of the series below is convergent. You should name any result or test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n^2+2}$ [6]

Question 8

This question concerns the symmetry group G of the regular hexagon shown below.



Let $g \in G$ be the clockwise rotation of the hexagon through $\frac{\pi}{3}$ about its centre, and let $h \in G$ be the reflection of the hexagon in the line through the vertices at locations 2 and 5.

(a) Write g, g^2 and h in cycle form, using the numbering of the locations of the vertices as shown above.

(b) Express the conjugate ghg^{-1}, of h by g, in cycle form and identify this conjugate as a symmetry of the hexagon.

(c) Are the symmetries $(14)(25)(36)$ and $(12)(36)(45)$ conjugate as symmetries in G? Justify your answer briefly. [5]

Question 9

The set $G = \{1, 3, 7, 9, 11, 13, 17, 19\}$ forms a group under multiplication modulo 20. (You are NOT asked to prove this statement.)

(a) Show that $H = \{1, 11\}$ is a subgroup of G.

(b) Write down the left cosets of H in G.

(c) Explain why H is a normal subgroup of G.

(d) Write down a group from the course that is isomorphic to the quotient group G/H. Give a brief reason for your answer. [6]

Question 10

In this question $\mathbb{C}^*$ is the group of non-zero complex numbers under multiplication, and ϕ is the function defined by

$$\phi : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$
$$z \longmapsto |z|^2.$$

(a) Prove that ϕ is a homomorphism.

(b) Determine the kernel and image of ϕ, and identify the quotient group $\mathbb{C}^* / \text{Ker}(\phi)$ up to isomorphism. (That is, give a group from the course that is isomorphic to $\mathbb{C}^* / \text{Ker}(\phi)$). [5]

Question 11

(a) Find an affine transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which sends the points $(0, 0)$, $(1, 0)$, $(0, 1)$ to the points $(2, 6)$, $(1, 8)$, $(5, 1)$, respectively.

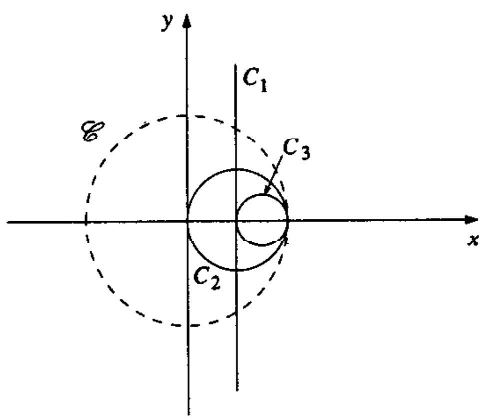
(b) Find the inverse of t, expressing your answer in the form $t^{-1}(\mathbf{x}) = \mathbf{Ax} + \mathbf{b}$.

(c) Check your answer by evaluating $t^{-1}(1, 8)$. [5]

Question 12

In the following diagram:

C_1 is the extended line with equation $x = \frac{1}{2}$;
C_2 is the circle of radius $\frac{1}{2}$ centred at $(\frac{1}{2}, 0)$;
C_3 is the circle of radius $\frac{1}{4}$ centred at $(\frac{3}{4}, 0)$.



On a single diagram, sketch the images of C_1, C_2 and C_3 under inversion in the unit circle $\mathscr{C}$. Mark clearly which image is which. [5]

Question 13

Determine whether the function f, defined below, is differentiable at 0.

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} x + \sin x, & x \le 0, \\ x^2 + 2x, & x > 0. \end{cases}$$ [5]

Question 14

Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n(-2)^n}.$$ [6]

PART II

(i) *You should attempt* **no more than THREE** *questions from this part.*

(ii) *Each question carries 10 marks. The mark allocation for each section of a question is given in square brackets beside the section.*

(iii) *Start each question on a new page of your answer book.*

Question 15

The Cayley table for a group G is shown below.

	e	a	b	c	d	f	g	h
e	e	a	b	c	d	f	g	h
a	a	c	h	d	g	b	f	e
b	b	h	g	e	a	d	c	f
c	c	d	e	g	f	h	b	a
d	d	g	a	f	b	e	h	c
f	f	b	d	h	e	c	a	g
g	g	f	c	b	h	a	e	d
h	h	e	f	a	c	g	d	b

(a) Show that G is the cyclic group generated by the element a. [2]

(b) Determine the possible orders for subgroups of G. [2]

(c) Give an example of a subgroup of G for each of the possible orders in your answer to part (b). [4]

(d) Give brief reasons why G is not isomorphic to either $S(\square)$ or $\mathbb{Z}_6$. [2]

Question 16

This question concerns the matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

(a) Find the eigenvalues of $\mathbf{A}$. [3]

(b) Find the eigenspaces of $\mathbf{A}$. [3]

(c) Find an orthonormal eigenvector basis of $\mathbf{A}$. [2]

(d) Write down an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D}$. [2]

Question 17

Determine whether the following sequences $\{a_n\}$ are convergent, stating the limit of the sequence if it exists. You should name any result or test that you use.

(a) $a_n = \dfrac{n^2 + 2(n!)}{n! - 2^n}, \quad n = 1, 2, \ldots$ [3]

(b) $a_n = \dfrac{3^n + n^2}{n^3 + 2^n}, \quad n = 1, 2, \ldots$ [3]

(c) $a_n = \dfrac{1 + 2n + (-1)^n n^2}{n^2 + 2}, \quad n = 1, 2, \ldots$ [4]

Question 18

This question concerns the set of matrices

$$U = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}.$$

(a) Show that U forms a group under matrix multiplication. [4]

The group U acts on the plane $\mathbb{R}^2$ as follows:

$$\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \wedge (x, y) = (x + by, ay)$$

for each $(x, y) \in \mathbb{R}^2$.

(You are NOT asked to verify that this is a group action. Essentially the action is matrix multiplication but you are expected to work with the group action on the plane.)

(b) Find the orbit of

(i) $(1, 0)$

(ii) $(0,1)$.

Give a geometric description of each of these orbits and deduce the set of orbits of the action. [4]

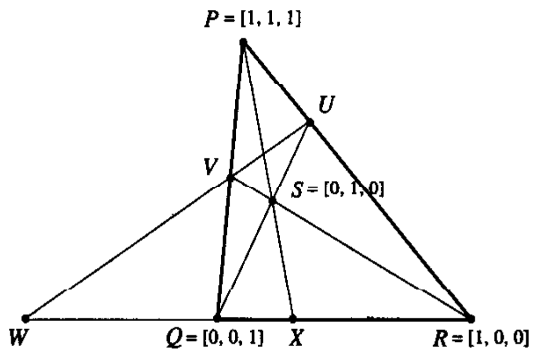
(c) Find the stabiliser of

(i) $(1, 0)$

(ii) $(0, 1)$. [2]

Question 19

In the following figure PQR is a triangle, in which the vertices P, Q and R have homogeneous coordinates $[1, 1, 1]$, $[0, 0, 1]$ and $[1, 0, 0]$, respectively. The point S has homogeneous coordinates $[0, 1, 0]$.



(a) Determine the homogeneous coordinates of:

(i) the Point X where QR meets PS;

(ii) the Point U where RP meets QS;

(iii) the Point V where PQ meets RS;

(iv) the Point W where UV meets QR. [6]

(b) Calculate the cross-ratio $(QRXW)$. [2]

(c) Now suppose that, in the above diagram, $RX = 2$ and $XQ = 1$. Use your answer to part (b) to calculate the distance QW. [2]

Question 20

This question concerns the linear flow for which the velocity function is

$$V(x, y) = (3x + y, 5x - y).$$

(a) Write down

(i) the matrix A of the flow;

(ii) the first order differential equations satisfied by the co-ordinate functions f and g of any flow function $\alpha = (f, g)$ for this flow;

(iii) a second order differential equation satisfied by both f and g. [3]

(b) Find the general solution of the differential equation in part (a)(iii). [2]

(c) Find the general form of the flow function α for V. [2]

(d) Determine the flow function α for V that satisfies $\alpha(0) = (1, 1)$. [2]

(e) Use your answer to part (d) to write down the equation of one barrier line of the flow. [1]

[END OF QUESTION PAPER]