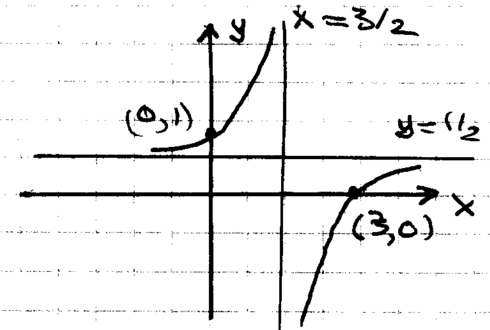
M203 1997 Examination

1. (a) asymptotes $x = 3/2$ (bottom line 0)
$y = \frac{1}{2}$ (as $x \to \pm\infty$)

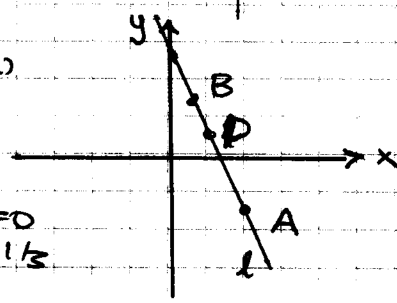


(b) $f(0) = 1$ so $(0,1)$ on graph
$y = 0$ when $x = 3$, so $(3,0)$ on graph.

(Shape is like $y = -1/x$)

2. (b) $\underline{r} = \lambda \underline{a} + (1-\lambda)\underline{b}$
$= \lambda(4,-3) + (1-\lambda)(1,3) = \underline{(3\lambda+1,\ 3-6\lambda)}$

(a)



(c) ℓ has direction $\vec{AB} = \underline{b} - \underline{a} = (-3,6)$
$\underline{r} \perp \ell \iff \underline{r}.(-3,6) = 0$, ie $-9\lambda - 3 + 18 - 36\lambda = 0$
ie $45\lambda = 15$ so $\lambda = 1/3$
P is $(2,1)$ (putting $\lambda = 1/3$ in "$\underline{r}$")

3. (a) $\begin{pmatrix} 1 & 3 & -2 \\ 2 & 1 & 1 \\ 1 & -7 & 8 \end{pmatrix} \underset{R_3 - R_1}{\overset{R_2 - 2R_1}{\sim}} \begin{pmatrix} 1 & 3 & -2 \\ 0 & -5 & 5 \\ 0 & -4 & 4 \end{pmatrix} \overset{-\frac{1}{5}R_2}{\sim} \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & -4 & 4 \end{pmatrix} \underset{R_3 + 4R_2}{\overset{R_1 - 3R_2}{\sim}} \underline{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}}$ (row reduced)

(b) System reduces to $\begin{cases} x \quad + z = 0 \\ \quad y - z = 0 \\ \quad 0 = 0 \end{cases}$ Solution set $\{(-z, z, z) : z \in \mathbb{R}\}$

4. (a) $A = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$ (by inspection)

(b) $B = AP = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = \underline{\begin{pmatrix} 5 & 0 \\ 1 & 7 \end{pmatrix}}$ Transition matrix for $\{(1,-2), (2,1)\}$ is $P = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

(c) $C = P^{-1}AP = P^{-1}B = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 1 & 7 \end{pmatrix}$
$= \frac{1}{5}\begin{pmatrix} 3 & -14 \\ 11 & 7 \end{pmatrix} = \underline{\begin{pmatrix} 3/5 & -14/5 \\ 11/5 & 7/5 \end{pmatrix}}$.

5. If $s \in E$, $s = 5 - 4/n^2 < 5$, so 5 is an upper bound

Suppose $t < 5$ $\quad 5 - 4/n^2 > t \iff 5 - t > 4/n^2$
$\iff n^2 > 4/5-t$
$\iff n > 2/\sqrt{5-t}$

[We show $t < 5$ is not an upper bound]

By the Archimedean Principle there exist such n
Thus t is not an upper bound for E
Hence 5 is the least upper bound.

6. (a) $a_n = \frac{n^2+2}{n+3n^2} = \frac{1+2/n^2}{1/n+3} \to \frac{1}{3}$ as $n \to \infty$ [Always worth asking is $\{a_n\}$ null!

ie $\{a_n\}$ is not null, so Σa_n diverges by non Null test

(b) $a_n = n^2 2^n / n!$ [2^n, $n!$ involved – Ratio

so $a_{n+1}/a_n = \frac{(n+1)^2 2^{n+1}}{(n+1)!} \cdot \frac{n!}{n^2 2^n} =$ [$n!/(n+1)! = 1/n+1$ $\quad 2^{n+1}/2^n = 2$

$= (n+1)^2 . 2 / (n+1) n^2 = 2(n+1)/n^2 \to 0$ as $n \to \infty$

As limit is $0 < 1$, Σa_n converges by Ratio Test.

7. Let $g(x) = 2e^{x} - 1$, $h(x) = \cos 2x$. These are continuous as they are combinations of basic continuous functions.

For $c < 0$, $f(x) = g(x)$ on $]-\infty, 0[$, so f is continuous at c (Local Rule)
For $c > 0$, $f(x) = h(x)$ on $]0, \infty[$, so f is continuous at c (Local Rule)

On $]-\infty, \infty[$ $\quad f(x) = g(x) \quad x < 0$
$\quad f(x) = h(x) \quad x > 0$
$\quad f(0) = g(0) = h(0) = 1$
So f continuous at 0 by Glue Rule.
Hence f is continuous on $\mathbb{R}$ (ie at every point of $\mathbb{R}$).

8. (a) $g = (1\,4\,7\,2\,5\,8\,3\,6)$ $\qquad$ (3/8 of a revolution)

(b) $h = (1\,5)(2\,4)(6\,8)$

(c) $ghg^{-1} = (g(1)\,g(5))(g(2)\,g(4))(g(6)\,g(8))$ (Handbook Method)
$= (4\,8)(5\,7)(1\,3)$.

This is reflection in line through 2 and 6

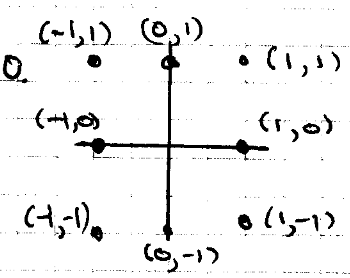
(d) No – the first is a reflection, the second a half turn

9. (a) $\langle 4 \rangle = \{4, 16, 10, 13, 25, 19, 22, 7, 1\}$ $\qquad$ (try elements to find a generator) – first time lucky!
so 4 is a generator
and hence G is cyclic.

(b) Suppose ϕ is a homomorphism from G onto a group H – so $\mathrm{Im}(\phi) = H$
By the Correspondence Theorem $|\mathrm{Im}(\phi)| = |H|$ must divide $|G| = 9$
$|\mathbb{Z}_4| = 4$, $|\mathbb{Z}_6| = 6$ so these cannot arise
$|\mathbb{Z}_3| = 3$ and $3|9$ so this is possible

(c) Again by the Correspondence Theorem, $|\mathrm{Ker}\,\phi|\,|\mathrm{Im}\,\phi| = |G|$, so $\phi: G \to \mathbb{Z}_3$ must have $|\mathrm{ker}(\phi)| = 3$
Since G is cyclic, its subgroups are cyclic – the only one of order 3 is $\{10, 19, 1\}$.
ie $\mathrm{ker}(\phi) = \{10, 19, 1\}$.

10.



(a) By applying each element of $S(\square)$ to the points we get $\qquad$ [Sketch not needed but useful
$\mathrm{Orb}(1,0) = \{(1,0), (0,1), (-1,0), (0,-1)\}$
$\mathrm{Orb}(1,-1) = \{(1,-1), (1,1), (-1,-1), (-1,1)\}$

(b) Checking effect of each element on the given point
$\mathrm{Stab}(1,0) = \{e, q_0\}$
$\mathrm{Stab}(1,-1) = \{e, q_{3\pi/4}\}$

Note: the Orbit Stabilizer Theorem shows that each "Stab" has order 2

11. (a) $M(z) = K\dfrac{z-4}{z-(-2+4i)} = K\dfrac{z-4}{z+2-4i}$ (Handbook)

where $1 = M(4+4i) = \dfrac{K.4i}{6}$ so $K = \dfrac{6}{4i} = -\dfrac{3}{2}i$

ie. $\underline{M(z) = \left(-\dfrac{3}{2}i\right)\dfrac{(z-4)}{(z+2-4i)}}$

(b) M maps $\mathcal{E}$ through $4, 4+4i, -2+4i$ to $\hat{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$

$M(-1-i) = \left(-\dfrac{3}{2}i\right)\left(\dfrac{-5-i}{1-5i}\right) = \dfrac{3i}{2}\dfrac{(5+i)(1+5i)}{26} = -\dfrac{3}{2} \in \mathbb{R}$

Thus $(-1-i)$ lies on $\mathcal{E}$.

$M(\infty) = -\dfrac{3i}{2}$ so ∞ is not on $\mathcal{E}$, ie $\mathcal{E}$ is a circle

12. By Handbook Strategy $A = \begin{pmatrix} \lambda & \mu & \nu \\ \lambda & 2\mu & 0 \\ -\lambda & -\mu & -3\nu \end{pmatrix}$ where $A\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\-6\end{pmatrix}$

ie $\begin{cases} \lambda + \mu + \nu = 2 & (1) \\ \lambda + 2\mu = -1 & (2) \\ -\lambda - \mu - 3\nu = -6 & (3) \end{cases}$ (1)+(3) gives $\nu = 2$ (1) $\lambda + \mu + 2 = 2$ (2) $\lambda + 2\mu = -1$ so $\lambda = 1, \mu = -1$

$$\underline{A = \begin{pmatrix} 1 & -1 & 2 \\ 1 & -2 & 0 \\ -1 & 1 & -6 \end{pmatrix}}$$

13. $f'(x) = -\frac{3}{2}(5-x)^{1/2}$ $\quad f(1) = 4^{3/2} = 8$

$f''(x) = \frac{3}{4}(5-x)^{-1/2}$ $\quad f'(1) = -3$

$f'''(x) = \frac{3}{8}(5-x)^{-3/2}$ $\quad f''(1) = 3/8$

$\underline{T_2(x)} = 8 - 3(x-1) + \frac{3}{8.2!}(x-1)^2 = \underline{8 - 3(x-1) + \frac{3}{16}(x-1)^2}$

On $[1,2]$ $|f'''(c)| \le \frac{3}{8}(5-2)^{-3/2} = \frac{3}{8.3\sqrt{3}} = 1/8\sqrt{3}$ ($c=4$ gives smaller value)

So (Handbook) $\underline{\text{error}} \le \frac{1}{3!8\sqrt{3}}(2-1)^3 = \frac{1}{48\sqrt{3}} \underline{< \frac{1}{48}}$, as required

14. (a) $I_0 = \int_1^e x^4\,dx = \left[\frac{1}{5}x^5\right]_1^e = \frac{1}{5}e^5 - \frac{1}{5}$

(b) Integrating by parts with $f(x) = (\log x)^n$, $g'(x) = x^4$

$I_n = \left[\frac{1}{5}x^5(\log x)^n\right]_1^e - \int_1^e \frac{1}{5}x^5\, n(\log x)^{n-1}.\frac{1}{x}\,dx$

$= \frac{1}{5}e^5 - \frac{n}{5}\int_1^e x^4(\log x)^{n-1}\,dx = \underline{\frac{1}{5}e^5 - \frac{n}{5}I_{n-1}}$ $\begin{cases}\log e = 1\\ \log 1 = 0\end{cases}$

(c) $I_1 = \frac{1}{5}e^5 - \frac{1}{5}I_0 = \frac{1}{5}e^5 - \frac{1}{5}\left(\frac{1}{5}e^5 - \frac{1}{5}\right)$ (put $n=1$

$= \frac{4}{25}e^5 + \frac{1}{25}$

$I_2 = \frac{1}{5}e^5 - \frac{2}{5}I_1 = \frac{1}{5}e^5 - \frac{2}{5}\left(\frac{4}{25}e^5 + \frac{1}{25}\right)$ ($n=2$

ie. $\underline{I_2 = \frac{17}{125}e^5 - \frac{2}{125}}$

mid-point

15. (a) $\underline{p = \frac{3}{5}a}$ (P is $\frac{3}{5}$ way along OA), $\underline{q = \frac{1}{2}(b + p) = \frac{3}{10}a + \frac{1}{2}b}$.

(b) AB : $r = \lambda a + (1-\lambda) b$ OQ : $s = \mu q + (1-\mu) 0$
$= \mu q = \mu \frac{3}{10} a + \mu \frac{1}{2} b$

Where these meet $\lambda a + (1-\lambda) b = \frac{3\mu}{10} a + \frac{\mu}{2} b$ ie $\begin{cases} \lambda = 3\mu/10 \\ 1-\lambda = \mu/2 \end{cases}$ so $\mu = 5/4$, $\lambda = 3/8$

Thus R has position vector $\underline{r = \frac{3}{8}a + \frac{5}{8}b}$ $\underline{\frac{AR}{RB} = \text{"}\frac{1-\lambda}{\lambda}\text{"} = \frac{5}{3}}$

(c) $R = \frac{3}{8}(11,-7) + \frac{5}{8}(-5,1) = (1,-2)$

$\underline{\cos\theta} = \frac{b \cdot r}{\|b\|\|r\|} = \frac{-7}{\sqrt{26}\sqrt{5}} = \underline{-7/\sqrt{130}}$.

16. (a) $0 \in S$ (put $a = b = 0$) so $\underline{S \text{ non-empty}}$
Let $u = (a, b, 2a+b, a-3b)$, $v = (c, d, 2c+d, c-3d)$ $(\in S)$
$u + v = ((a+c), (b+d), 2(a+c)+(b+d), (a+c)-3(b+d)) \underline{\in S}$.
For $\lambda \in \mathbb{R}$ $\lambda u = (\lambda a, \lambda b, 2(\lambda a)+(\lambda b), (\lambda a)-3(\lambda b)) \underline{\in S}$

Hence S is a subspace

(b)(i) $u = (1,1,3,-2) \in S$ $(a = b = 1)$, $v = (1,-1,1,4) \in S$ $(a = 1, b = -1)$

(ii) $(a, b, 2a+b, a-3b) = \alpha(1,1,3,-2) + \beta(1,-1,1,4) \iff \begin{cases} \alpha + \beta = a \\ \alpha - \beta = b \\ 3\alpha + \beta = 2a+b \\ -2\alpha + 4\beta = a - 3b \end{cases}$

First two give $\alpha = \frac{1}{2}(a+b)$, $\beta = \frac{1}{2}(a-b)$ and these satisfy 3rd and 4th

Thus $\underline{(a, b, 2a+b, a-3b) = \frac{1}{2}(a+b)(1\ 1\ 3\ {-2}) + \frac{1}{2}(a-b)(1\ {-1}\ 1\ 4)}$

(iii) Since u, v are linearly independent (non-parallel) and span S (by (ii)) $\{u, v\}$ is a basis for S.

(c) We want $w \in S$ with $u \cdot w = 0$. Say $w = (a, b, 2a+b, a-3b)$
$u \cdot w = 0 \iff a + b + 3(2a+b) - 2(a-3b) = 0$
i.e. $5a + 10b = 0$ eg $a = 2, b = -1$ (any solution will do)

$\underline{w = (2, -1, 3, 5)}$

(d) $(5,2,12,-1) = \alpha(1\ 1\ 3\ {-2}) + \beta(2\ {-1}\ 3\ 5) \iff \left.\begin{array}{l} \alpha + 2\beta = 5 \\ \alpha - \beta = 2 \\ 3\alpha + 3\beta = 12 \\ -2\alpha + 5\beta = -1 \end{array}\right\} \alpha = 3, \beta = 1$

so $\underline{(5,2,12,-1) = 3(1\ 1\ 3\ {-2}) + 1.(2,-1,3,5)}$

Could use strategy for orthogonal basis

17. (a) $a_n = \dfrac{3^n/n! + 1}{2 + 2^n/n!}$ [$n!$ dominant

so $a_n \to \frac{1}{2}$ as $n \to \infty$ since $\{c^n/n!\}$ is basic null and by the Combination Rules

(b) $\dfrac{1}{a_n} = \dfrac{2n + 2^n - 5}{3^n + n + 1}$ [observe that top line dominant so "guess" $a_n \to \infty$ Hence look at $1/a_n$

$= \dfrac{2n(\frac{1}{3})^n + (\frac{2}{3})^n - 5(\frac{1}{3})^n}{1 + n(\frac{1}{3})^n + (1/3)^n}$ [3^n dominant

$\to 0$ as $n \to \infty$ as $\{c^n\}, \{nc^n\}$ basic null for $|c| < 1$ and using Combination Rules

$a_n > 0$ for $n > 1$, so "eventually positive"

Hence $a_n \to \infty$ by Reciprocal Rule

Thus $\{a_n\}$ is not convergent (Boundedness)

(c) For n even $a_n = \dfrac{n^2}{2n^2+n+1} = \dfrac{1}{2+\frac{1}{n}+\frac{1}{n^2}}$ [dominant term is n^2 but $(-1)^n$ causes problems

$\to \frac{1}{2}$ by Combination Rules as $\{1/n^p\}$ basic null.

For n odd $a_n = \dfrac{-n^2}{2n^2+n+1} \to -\frac{1}{2}$, as above.

Thus $\{a_n\}$ is divergent, by Subsequence Rule.

18. [A really horrible question]

(a) From table $\langle b \rangle = \langle b, d, g, e \rangle$

(b) Extracting entries from table we get K is closed, contains e and each element is its own inverse, so K a subgroup

	e	d	i	m
e	e	d	i	m
d	d	e	m	i
i	i	m	e	d
m	m	i	d	e

(c) $eH = H = He$

$aH = \{c, f, h, a\} = Ha$ (using the table!)

$iH = \{k, m, o, i\} = Hi$

$jH = \{l, n, p, j\} = Hj$ Since left/right cosets agree H normal

$aK = \{a, f, o, l\}$, $Ka = \{a, f, j, n\} \neq aK$, so K not normal

(d) $Q = \{eH, aH, iH, jH\}$ (the distinct cosets)

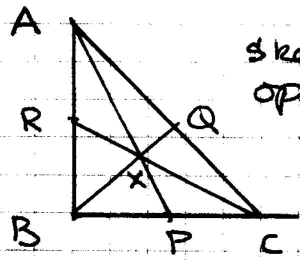
(e) $eH * eH = aH * aH = iH * iH = jH * jH = H$ [order 4 so cyclic or Klein] recall that $xH * yH = (xy)H$

ie all elements of Q are of order 2

Hence Q is a Klein group (ie S(rectangle))

19. $A(0,1)$ so $R(0, \frac{1}{2})$
$B(0,0)$
$C(1,0)$ so $P(\frac{1}{2}, 0)$

$Q = (\frac{1}{2}, \frac{1}{2})$ $(= \frac{1}{2}(\underline{a} + \underline{c}))$



sketch optional

(a) $AP: y = -2x + 1$
$CR: y = -\frac{1}{2}x + \frac{1}{2}$

by inspection or otherwise

At X $\begin{cases} y = -2x+1 \\ y = -\frac{1}{2}x + \frac{1}{2} \end{cases}$ $x = \frac{1}{3}$, $y = \frac{1}{3}$ ie $\underline{X = (1/3, 1/3)}$

(b) BQ is $y = x$ (obvious), so $X \in BQ$.

(c) Using the x-coordinates $\frac{AX}{XP} = \frac{1/3}{1/2 - 1/3} = \frac{2}{1}$, $\frac{BX}{XQ} = \frac{1/3}{\frac{1}{2} - \frac{1}{3}} = \frac{2}{1}$, $\frac{CX}{XR} = \frac{\frac{1}{3} - \frac{1}{2}}{0 - \frac{1}{3}} = \frac{2}{1}$

ie $\underline{\frac{AX}{XP} = \frac{BX}{XQ} = \frac{CX}{XR} = \frac{2}{1}}$

(d) We can apply an affine transformation t to map D to A, E to B, F to C

Since t preserves mid-points, it maps medians to medians
As above, the medians concur at a point two thirds along each.
Since t^{-1} also preserves ratios, this is also true for $\triangle DEF$

20. (a) (i) $A = \begin{pmatrix} -7 & 2 \\ 1 & -8 \end{pmatrix}$ $\quad tr(A) = -15$, $\det(A) = 54$

(ii) $f' = -7f + 2g$ (1)
$g' = f - 8g$

(iii) $f'' + 15f' + 54f = 0$

It's all in the Handbook!

(b) Auxiliary equation $\lambda^2 + 15\lambda + 54 = 0$ ie $(\lambda + 6)(\lambda + 9) = 0$
Roots $\lambda = -6, -9$ (real, distinct)

New bit of question

Barrier lines are "eigenlines" $\lambda = -6$: $\begin{matrix} -x + 2y = 0 \\ (x - 2y = 0) \end{matrix}$, $\lambda = -9$: $\begin{matrix} 2x + 2y = 0 \\ (x + y = 0) \end{matrix}$

ie lines are $\underline{x = 2y}$ and $\underline{x + y = 0}$

(c) $\underline{f(t) = ce^{-6t} + de^{-9t}}$

(d) $g(t) = \frac{1}{2}(f'(t) + 7f(t))$ (from (1))
$= \frac{1}{2}(-6ce^{-6t} - 9de^{-9t} + 7ce^{-6t} + 7de^{-9t})$
$= \frac{1}{2}ce^{-6t} - de^{-9t}$

$\underline{\alpha(t) = (f(t), g(t))}$ with f, g as above

(e) $(1, 1) = \alpha(0) = (f(0), g(0)) = (c + d, \frac{1}{2}c - d)$

ie $\left.\begin{matrix} c + d = 1 \\ \frac{1}{2}c - d = 1 \end{matrix}\right\}$ $c = 4/3$, $d = -1/3$

$\underline{\alpha(t) = (\frac{4}{3}e^{-6t} - \frac{1}{3}e^{-9t}, \frac{2}{3}e^{-6t} + \frac{1}{3}e^{-9t})}$