PART II

(i) *You should attempt* **not more than FOUR** *questions from this part.*

(ii) *Each question carries 10 marks. The mark allocation for sections of questions is given in square brackets beside each section.*

(iii) *Start each question on a new page of your answer book.*

(iv) *The questions are grouped together according to subject area.*

LINEAR ALGEBRA

Question 13

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 3 & 1 \\ 1 & 5 & -3 \end{pmatrix}$ represents a linear transformation $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with respect to the standard basis in both the domain and the codomain.

(a) Find the row-reduced form of the matrix $\mathbf{A}$. [2]

(b) Hence determine the kernel of f. [2]

(c) Determine the dimension of $\text{Im}(f)$, and find a basis for $\text{Im}(f)$. [3]

(d) Determine whether the vector $\mathbf{v} = (2, 3, 4)$ belongs to $\text{Im}(f)$. [3]

Question 14

In this question V denotes the vector space of functions given by $V = \{f : f(x) = ae^x + be^{-x} + c;\ a, b, c \in \mathbb{R}\}$. (You are NOT asked to prove that V is a vector space.)

(a) Determine whether each of the following subsets of V is a subspace of V.

(i) $S = \{f \in V : f(x) = a(e^x - e^{-x}), a \in \mathbb{R}\}$

(ii) $T = \{f \in V : f(0) = 2\}$ [5]

(b) Show that the function

$$t : V \longrightarrow \mathbb{R}^2$$
$$t : f \longmapsto (a - b, a + c), \quad \text{where } f(x) = ae^x + be^{-x} + c,$$

is a linear transformation, and determine the kernel of t in the form

$$\text{Ker}(t) = \{f \in V : f(x) = ae^x + be^{-x} + c;\ \text{some conditions on } a, b, c\}.$$ [5]

ANALYSIS A

Question 15

Determine whether each of the following sequences $\{a_n\}$ is convergent, stating the limit of the sequence (if a limit exists). You should state any result or test that you use.

(a) $a_n = \dfrac{n! + 2^n}{n^2 + 3(n!) + 1}, \quad n = 1, 2, \ldots$ [3]

(b) $a_n = \dfrac{n^2 + 4^n - 4}{n^3 + 3^n - 5}, \quad n = 1, 2, \ldots$ [3]

(c) $a_n = \dfrac{(-1)^n n^3}{4n^3 + n + 1}, \quad n = 1, 2, \ldots$ [4]