Question 16

(a) Determine the set of points at which the following function is continuous.

$$f(x) = \begin{cases} 1 + 2x + x^2, & x \le 0, \\ e^x + x, & x > 0. \end{cases}$$ [6]

(b) Let p be the polynomial defined by

$$p(x) = x^5 + 2x^4 - 2x^3 - 2.$$

(i) Show that all real zeros of p lie in $]-3, 3[$.

(ii) Prove that p has at least three real zeros. You may use the fact that $p(-3) = -29$ and $p(3) = 349$. [4]

GROUP THEORY

Question 17

This question concerns the subset X of $\mathbb{R}^2$ where

$X = \{(a, b) : b \neq 0\}$ (that is, $\mathbb{R}^2$ excluding the x-axis)

and the binary operation $*$ defined by

$$(a, b) * (c, d) = (ad + bc, bd).$$

(a) Verify that the element $(0, 1)$ acts as the identity element in X. [1]

(b) Find a formula for the inverse of the element $(a, b) \in X$, and hence show that $(X, *)$ satisfies the inverses axiom G3. [2]

In parts (c), (d) and (e) below you may assume that $(X, *)$ is a group: you are NOT asked to check the remaining group axioms G1 and G4.

(c) Prove that

$$\phi : (X, *) \longrightarrow (\mathbb{R}, +)$$
$$\phi : (a, b) \longmapsto a/b$$

is a homomorphism. [2]

(d) Find the kernel of the homomorphism ϕ defined in part (c). [2]

(e) Justify the statement that the quotient group

$$X/\{(0, b) : b \in \mathbb{R}, b \neq 0\}$$

exists, and show that it is isomorphic to $(\mathbb{R}, +)$. [3]

Question 18

The group $G = \{e, r_{2\pi/3}, r_{4\pi/3}, q_{\pi/6}, q_{\pi/2}, q_{5\pi/6}\}$ (which is isomorphic to $S(\Delta)$) acts on the plane $\mathbb{R}^2$ in the natural way; that is, for $g \in G$ and $(a, b) \in \mathbb{R}^2$, we define the action $\wedge$ by the formula

$$g \wedge (a, b) = g((a, b)).$$

(You are NOT asked to justify any of these results.)

(a) Find the orbit of each of the points

(i) $(0, 0)$; (ii) $(0, 1)$; (iii) $(1, 0)$. [5]

(b) Find the stabiliser of each of the points

(i) $(0, 0)$; (ii) $(0, 1)$; (iii) $(1, 0)$. [2]

(c) For each $g \in G$ specify $\mathrm{Fix}(g)$ as a subset of the plane. [3]