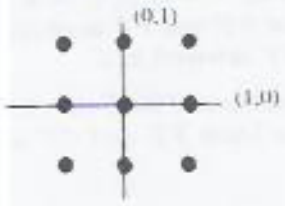
Question 18

The set $X = \{(m,n) : m,n \in \{-1,0,1\}\}$ is shown below.



Let $\mathbf{P} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The group of matrices

$$G = \{\mathbf{I}, \mathbf{P}, \mathbf{P}^2, \mathbf{P}^3, \mathbf{Q}, \mathbf{PQ}, \mathbf{P}^2\mathbf{Q}, \mathbf{P}^3\mathbf{Q}\}$$

is isomorphic to

$$S(\square) = \{e, a, b, c, r, s, t, u\}.$$

(You are NOT asked to justify this result.)

An action of G on X is defined by $g \wedge (m,n) = (k,l)$ if $g\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} k \\ l \end{pmatrix}$, for any $g \in G$ and $(m,n) \in X$.

(a) By considering $\mathbf{P}_\wedge(m,n)$ and $\mathbf{Q}_\wedge(m,n)$ verify that axiom $GA1$ holds. [2]

In parts (b) and (c) below you may assume that G acts on X.

(b) Find the orbit and the stabilizer of each of the points:

(i) $(1,-1)$,

(ii) $(0,0)$,

(iii) $(-1,0)$. [5]

(c) Calculate $|\mathrm{Fix}(g)|$ for each $g \in G$ and use the Counting Theorem to show that there are exactly three orbits. [3]

GEOMETRY

Question 19

In this question C is the conic with equation

$$x^2 - 2\sqrt{3}xy + 3y^2 + 3\sqrt{3}x + y + 3 = 0.$$

To help you to reduce the equation of C to standard form, you are given the following information. The eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}$$

are 0 and 4 and corresponding eigenvectors are $(\sqrt{3},1)$ and $(1,-\sqrt{3})$.

(a) Write the equation of C in matrix form. [1]

(b) Write down a matrix $\mathbf{P}$ such that $\mathbf{P}^T\mathbf{AP} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$. [2]

(c) Find the equation of the conic after the transformation $\mathbf{x} \mapsto \mathbf{P}^T\mathbf{x}$, (i.e. $\mathbf{x} = \mathbf{Px}'$). DO NOT leave your answer in matrix form. [3]

(d) Find and specify a translation that reduces the equation of part (c) to standard form. Hence classify the conic. [4]