Question 10

(i) Let h be the function

$$\mathrm{Pr}[\mathrm{Cn}[s, z], \mathrm{Cn}[\exp, \mathrm{Cn}[s, \mathrm{id}_3^3], \mathrm{id}_2^3]]$$

where exp is the function defined by $\exp(x, y) = x^y$.

Compute the values of

(a) $h(3, 0)$,

(b) $h(3, 2)$. [4]

(ii) In this part you may present your arguments using either formal or informal definitions.

(a) Show that the function dif, where

$$\mathrm{dif}(x, y) = \begin{cases} x - y & \text{if } x \geq y, \\ 0, & \text{if } x < y. \end{cases}$$

is primitive recursive by defining it in terms of the initial functions. [2]

(b) Show that the functions sg and $\overline{\mathrm{sg}}$, where

$$\mathrm{sg}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases} \quad \text{and} \quad \overline{\mathrm{sg}}(x) = \begin{cases} 0, & \text{if } x > 0, \\ 1, & \text{if } x = 0, \end{cases}$$

are primitive recursive by defining them in terms of the initial functions and, if you wish, the function dif. [2]

(iii) Write down the pairs (x_1, x_2) of natural numbers for which $\mathrm{Mn}[f](x_1, x_2)$ is defined, where f is the function defined by

$$f(x_1, x_2, y) = x_1 \cdot y + ((x_2 \dot{-} x_1) \dot{-} y).$$

[3]

Question 11

(i) A Turing machine has a configuration with left number 26 and right number 59. Draw the configuration which results when the scanning head has moved one square to the right, and find its left and right numbers. [3]

(ii) In parts (a), (b) and (c) below, you may use any of the recursive functions, or results about them, given in the Logic Handbook without proving that they are recursive. You may give your answers as informal definitions.

(a) Show that the condition "$x \geq y$" on pairs (x, y) of natural numbers is primitive recursive. [3]

(b) Show that the function Min defined by

$$\mathrm{Min}(x, y) = \begin{cases} y, & \text{if } x \geq y, \\ x, & \text{if } x < y, \end{cases}$$

is primitive recursive. [$1\frac{1}{2}$]

(c) Show that the function f defined by

$$f(x_1, x_2, x_3) = \begin{cases} 2x_2, & \text{if } x_1 + 12 \leq \mathrm{Min}(x_2, x_3), \\ x_1^{x_3}, & \text{if } 2x_1 + 3x_2 + 4x_3 = 80, \\ 5, & \text{otherwise,} \end{cases}$$

is primitive recursive. [$3\frac{1}{2}$]