Question 14

(i) For each of the following terms, write down whether it is freely substitutable for x in the formula

$$\forall y(\forall x \exists t(x+y)=t \vee \exists z(z+t)=x).$$

(a) $(t+x)$

(b) $(x \cdot y)$

(c) $(x+z)$

Answer YES or NO in each case. [2]

(ii) (a) Give a formal proof to show that

$$\exists x(-\phi \,\&\, \psi), \forall x((\psi \,\&\, \theta) \to \phi) \vdash -\forall x \theta$$

where the variable x does not occur free in ϕ. Indicate the step(s) of your proof which require the condition on ϕ. [6]

(b) Hence, or otherwise, give a formal proof to show that

$$\forall x((\psi \,\&\, \theta) \to \phi),\ \forall x \theta \vdash -\exists x(-\phi \,\&\, \psi)$$

where the variable x does not occur free in ϕ. [3]

Question 15

For each of the following sentences, decide whether or not it is a theorem of Q. If it is a theorem of Q, write down a formal proof showing this. If it is not a theorem of Q, justify this. (*You may use without proof the fact that all the axioms of Q are true under the interpretations N^* and N^{**} given in the Logic Handbook.*)

(i) $\forall x((\mathbf{0} \cdot x) \cdot \mathbf{0}) = (x \cdot (\mathbf{0} + \mathbf{0}))$

(ii) $\forall x \exists y(x'' + y) = (y'' + x)$

(iii) $\forall x \forall y(x'' + y) = (y' + x')$ [11]

Question 16

(i) Explain briefly the meaning of *each* of the following statements about a theory T.

(a) T is complete;

(b) T is consistent;

(c) T is axiomatizable. [5]

(ii) Is $\mathbf{0} = \mathbf{1}$ a theorem of the theory Z of elementary Peano arithmetic? Explain your answer. [2]

(iii) Is the theory Z

(a) complete, (b) axiomatizable.

In each case, answer YES or NO. [1]

(iv) Which theorem(s) of the course answer(s) the question: *Is arithmetic axiomatizable?*

Explain why the theorem (or theorems) chosen answer(s) the question. (*Your answer may include references to any of the theorems listed in the Logic Handbook.*) [3]

[END OF QUESTION PAPER]