Answer as many questions as you wish. Full marks may be obtained by **complete** *answers to* **NINE** *questions, provided that no more than* **SIX** *questions have been selected from any one part. All questions carry equal marks.*

PART I NUMBER THEORY

Question 1

(i) Prove by induction that the formula

$$1+6+15+\cdots+n(2n-1)=\tfrac{1}{6}n(n+1)(4n-1)$$

holds for all $n \geq 1$. [4]

(ii) Use the Euclidean Algorithm to determine the greatest common divisor of 238 and 140. [2]

(iii) For which integer values of c does the linear Diophantine equation

$$51x-36y=c.$$

have solutions? Find the general solution of this equation, by any method, for the case when c takes the least positive value for which solutions exist. [5]

Question 2

(i) Prove that there are infinitely many primes of the form $4k+3$. [5]

(ii) Consider the sequence of integers

$$5,\ 8,\ 13,\ 20,\ 29,\ \ldots,$$

whose nth term $a_n=n^2+4$. Consecutive terms of this sequence are either relatively prime or are both divisible by a certain prime p. Determine this prime p and prove that, for all $n \geq 1$, $\gcd(a_n, a_{n+1})=1$ or p. [6]

Question 3

(i) Suppose that a and b are integers, d and m are positive integers and that d is a divisor of m. For each of the following, decide whether it is true or false. If true, prove it from the definition of congruence alone; if false, give a counterexample.

(a) If $a \equiv b \pmod{d}$ then $a \equiv b \pmod{m}$.

(b) If, for some integer k, $ka \equiv kb \pmod{m}$ then $a \equiv b \pmod{m}$.

(c) If $da \equiv db \pmod{m}$ then $a \equiv b \left(\mathrm{mod}\ \dfrac{m}{d}\right)$.

(d) If $a \equiv db \pmod{m}$ then $a \equiv 0 \pmod{d}$. [6]

(ii) Find the least positive integer which satisfies each of the following linear congruences simultaneously.

$$x \equiv 0 \pmod{3}; \quad x \equiv 4 \pmod{5}; \quad x \equiv 8 \pmod{11}.$$ [5]