$prod(x,y) = x \cdot y$

(10) (i) $h = Pr[Cn[s,z],\ Cn[prod, Cn[s, id_2^3],\ id_3^3]]$
Compute $h(3,0)$, $h(3,2)$.

Recall $h = Pr[f,g]$ means $h(x,0) = f(x)$
$h(x, s(y)) = g(x, y, h(x,y))$

thus $h(3,0) = f(3) = 1$

$h(3,2) = h(3, s(1)) = g(x, 1, h(x,1))$ $y=1$

$h(3,1) = h(3, s(0)) = g(3, 0, \overbrace{h(3,0)}^{1})$
$= 1 \cdot 1 = 1$

$h(3,2) = g(3,1,1)$
$= 2 \cdot 1 = 2.$

(ii) (a) Show prod is prim rec. by defining in terms of sum and initial fns.

$prod(x,0) = 0 = z(x)$
$prod(x, s(y)) = x \cdot (y+1) = xy + x$

$prod = Pr[z,\ sum(h(x,y), x)]$ ← informal.
$= Pr[z, Cn[sum, id_1^3, id_3^3]]$ formal

(b) Show $f(x,y) = (x+1)^y$ is prim rec. in terms of prod and initial fns.

$f(x,0) = (x+1)^0 = 1 = Cn[s,z]$
$f(x, y+1) = (x+1)^{y+1} = (x+1)^y \cdot (x+1) = f(x,y) \cdot (x+1)$
$\therefore g(x,y,f(x,y)) = Cn[prod, Cn[s, id_1^3], id_3^3]$

(iii) Write down the pairs (x_1, x_2) for which $Mn[f](x_1,x_2)$ is defined for $f(x_1, x_2, y) = (x_1+y)\cdot(x_2+y)$

$Mn[f] = \begin{cases} \text{the smallest } y \text{ for which } f(x_1,x_2,y) = 0 \\ \text{undefined otherwise} \end{cases}$

Let $f(x_1, x_2, y) = 0$; then $(x_1+y)\cdot(x_2+y) = 0$
$\therefore x_1 + y = 0$ or $x_2 + y = 0$

But $x_1 + y = 0 \Rightarrow x_1 = 0, y = 0$
$x_2 + y = 0 \Rightarrow x_2 = 0, y = 0$
Thus $Mn[f]$ defined for (x_1, x_2) with $x_1 = 0$ or $x_2 = 0$ or both.

(11) (i) Draw the configuration of a Turing machine tape with left number 23, right number 19

$23 = 10111$
$19 = 10011$

| 1 0 1 1 1 | 1 | 1 0 0 1 |

left numeral — right numeral in reverse

(ii) (We may use all results/fns in the Handbook)

(a) Show '$x \le y$' is prim recursive.

Let $f(x,y) = \overline{sg}(x \dot- y)$

then if $y > x$, $x \dot- y = 0$; $y = x$, $x \dot- y = 0$ } , $f(x,y) = \overline{sg}(0) = 1$
$y < x$, $x \dot- y > 0$ } $f(x,y) = 0$

$\therefore$ the characteristic fn of the cond^n is p.r. so the cond^n is prim. rec.

(b) Show $f(x,y) = \begin{cases} x \cdot y & \text{if } x \le y \\ x+y & x = 2y+1 \\ 3x & \text{otherwise} \end{cases}$ is p.r.

$f(x,y) = \begin{cases} g_1 & \text{if } C_1 \\ g_2 & \text{if } C_2 \\ g_3 & \text{otherwise} \end{cases}$

g_1, g_2, g_3 clearly p.r. fns^n
C_1, otherwise p.r. cond^{ns}
C_1, C_2, C_3 exhaustive

All that remains is to show:

C_1, C_2 mutually exclusive : if $x \le y$ then $x < y+1 \le 2y+1$

C_2 p.r. $Eq(x, 2y+1)$ is the characteristic fn.

this will do.

more formally $Cn[Eq, id_1^2, Cn[Cn[sum, id_2^2, id_2^2]$ … $sum, s(z(id_1^2))]$ … $Cn[s, Cn[z, …$