Question 13

(i) Show that the following formula takes truth value 1 under all interpretations of its symbols.

$((-\forall x\; x = 0 \;\&\; -(\exists x(x = 0 \leftrightarrow \forall x\; x = 0) \to \exists x\; x = 0)) \to (\forall x\; x = 0 \vee -\exists x\; x = 0))$ [3]

(ii) The following is a correct (but contorted) proof from which the assumption numbers have been omitted.

(1)	$\forall x(\theta \to -\psi)$	Ass
(2)	$(\phi \;\&\; \psi)$	Ass
(3)	$(\theta \to -\psi)$	UE, (1)
(4)	$(\theta \vee \phi)$	Taut, (2)
(5)	$\exists x(\phi \;\&\; \psi)$	Ass
(6)	$((\phi \;\&\; \psi) \to (\theta \vee \phi))$	CP, (4)
(7)	$(\phi \vee -\psi)$	Taut, (3), (4)
(8)	$\exists x(\phi \vee -\psi)$	EI, (7)
(9)	$\exists x(\phi \vee -\psi)$	EH, (8)

(a) Write down the assumptions in force on each line. [$2\frac{1}{2}$]

(b) Write down the tautology used on line (7). [$\frac{1}{2}$]

(c) For each of the following lines, write down whether the proof would necessarily still be correct were the line to be added to it.

(A)	2	(10) $\exists x(\theta \vee \phi)$	EI, (4)
(B)	2	(10) $\forall x(\theta \vee \phi)$	UI, (4)

Answer YES or NO. [2]

(iii) By giving an example from everyday maths along with a relevant rule of proof (in the Logic Handbook), explain why the validity of the rule you have chosen requires a variable not to occur free in some of the formulas mentioned within the statement of that rule. [3]

Question 14

(i) Which of the following terms are freely substitutable for x in the formula

$$\forall y(\exists x \forall t(x + t) = z \;\&\; \forall z(x + z) = t)?$$

(a) $(t + z)$

(b) $(x \cdot t)$

(c) $(x + y)$

Answer YES or NO in each case. [2]

(ii) Give formal proofs to establish each of the following results.

(a) $\exists x \exists y(x \cdot y) = y \vdash \exists y \exists x(x \cdot y) = y$ [3]

(b) $\phi \vdash (\forall x(\psi \vee -\phi) \to -\exists x - (\theta \to \psi))$ where the variable x does not occur free in ϕ. Indicate the step(s) of your proof which require the condition on ϕ. [6]

Question 15

For each of the following sentences, decide whether or not it is a theorem of Q. If it is a theorem of Q, write down a formal proof showing this. If it is not a theorem of Q, justify this. (*You may use without proof the fact that all the axioms of Q are true under the interpretations N^* and N^{**} given in the Logic Handbook.*)

(i) $\forall x(\mathbf{0} + (x \cdot \mathbf{0})) = ((\mathbf{0} + x) \cdot \mathbf{0})$

(ii) $\forall x \forall y -(x + y) = (x + y)''$

(iii) $\exists x -(x + x) = (x + x)''$ [11]