Answer as many questions as you wish. Full marks may be obtained by **complete** *answers to* **NINE** *questions, provided that no more than* **SIX** *questions have been selected from any one part. All questions carry equal marks.*

PART I NUMBER THEORY

Question 1

(i) Use the Euclidean Algorithm to determine the greatest common divisor of 161 and 253, and hence find integers x and y such that

$$\gcd(161, 253) = 161x + 253y.$$ [3]

(ii) The Fibonacci sequence is defined by

$$F_1 = F_2 = 1; \quad F_{n+2} = F_{n+1} + F_n, \quad \text{for } n \geq 1.$$

Use Mathematical Induction to prove that

$$F_1 + 2F_2 + 3F_3 + \cdots + nF_n = (n+1)F_{n+2} - F_{n+4} + 2$$

for all $n \geq 1$. [5]

(iii) Prove Euclid's Lemma, namely that if a, b and c are integers such that $a|bc$ with $\gcd(a, b) = 1$, then $a|c$. [You may assume, without proof, any other result given in the Handbook entry for Unit 1.] [3]

Question 2

For each of the following statements about integers k, m and n, decide whether it is true or false. If true, prove it: if false, justify your answer.

(i) Any number of the form $6k+1$, where $k \geq 1$, must have a prime divisor of this same form.

(ii) Any number of the form $6k+5$, where $k \geq 0$, must have a prime divisor of this same form.

(iii) If $\gcd(m, n) = 1$ then $\gcd(6m+1, 6n+1) = 1$.

(iv) There are infinitely primes of the form $6k+5$. [*11*]

Question 3

(i) Prove, from the definition of congruence alone, that if $na \equiv nb \pmod{mn}$ then $a \equiv b \pmod{m}$, where a, b, m and n are integers, with m and n positive. Use this result in solving the linear congruence

$$48x \equiv 12 \pmod{150}.$$ [5]

(ii) Solve the polynomial congruence $x^3 - 2x - 4 \equiv 0 \pmod{5}$. [2]

(iii) Find the least positive integer which satisfies each of the following linear congruences simultaneously:

$$x \equiv 1 \pmod{3}; \quad x \equiv 2 \pmod{5}; \quad x \equiv 2 \pmod{11}.$$ [*4*]

Question 4

(i) Use Fermat's Little Theorem (FLT) to show that $3^{100} - 2^{100}$ is divisible by 13. [*4*]

(ii) Determine the smallest prime divisor of $16! + 1$. [3]

(iii) Find the least positive remainder when 24^{24} is divided by 77. [Note that 77 is not prime.] [*4*]