Question 13

(i) Show that the following formula takes truth value 1 under all interpretations of its symbols.

$(((x = 0 \vee \forall x\, x = 0) \to \forall x(x = 0 \vee \forall x\, x = 0)) \to (-\forall x(x = 0 \vee \forall x\, x = 0) \to -x = 0))$ [3]

(ii) The following is a correct (but contorted) proof from which the assumption numbers have been omitted.

(1)	$(\psi \to \theta)$	Ass	1
(2)	$\forall x(\theta \,\&\, \phi)$	Ass	2
(3)	$(\theta \,\&\, \phi)$	UE, (2)	2
(4)	$\exists x(\psi \to \theta)$	Ass	4
(5)	$(\psi \to \theta)$	Taut, (1), (3)	1, 2
(6)	$\exists x(\psi \to \theta)$	EI, (5)	1, 2
(7)	$\exists x(\psi \to \theta)$	EH, (6)	2, 4
(8)	$((\psi \to \theta) \to \exists x(\psi \to \theta))$	CP, (6)	2
(9)	$((\psi \to \theta) \to \exists x(\psi \to \theta))$	Taut, (7)	2, 4

(a) Write down the assumptions in force on each line. [$2\frac{1}{2}$]

(b) Write down the tautology used on line (9). [$\frac{1}{2}$]

(c) For each of the following lines, write down whether the proof would still be correct were the line to be added to it.

(A) 1 (10) $\forall x(\psi \to \theta)$ UI, (1)
(B) 4 (10) $(\psi \to \theta)$ EH, (1)

Answer YES or NO. [2]

(iii) (a) Give an example of formulas ϕ and θ for which θ is a tautological consequence of ϕ. $x = x$, $x = x$ [$1\frac{1}{2}$]

(b) Give an example of formulas ϕ and θ for which θ is a logical, but not a tautological, consequence of ϕ. $x = x$, $\forall x\; x = x$ [$1\frac{1}{2}$]

(No justification is required in either case.)

Question 14

(i) Which of the following terms are freely substitutable for x in the formula

$$\forall t(\exists x(x + y) = z \to \exists y \forall x(x + t) = y)?$$

(a) $(x \cdot 0)'$

(b) $(x + t)$

(c) $(x + y)$

Answer YES or NO in each case. [2]

(ii) Give formal proofs to show each of the following results.

(a) $\exists x \forall y(x + y) = y \vdash \forall y \exists x(x + y) = y$ [3]

(b) $\forall x(-\phi \vee -\psi) \vdash (\psi \to -\exists x(\phi \,\&\, \theta))$, where the variable x does not occur free in ψ. Indicate the step(s) of your proof which require the condition on ψ. [6]