Question 5

(i) Prove that every number of the form $2^{k-1}(2^k - 1)$, where $2^k - 1$ is prime (k an integer with $k \geq 2$), is perfect. [3]

(ii) A positive integer n has the property that $n + 2\phi(n)$ is divisible by 3 (where ϕ is Euler's phi-function).

(a) Prove that n cannot be prime.

(b) Determine for which primes p, if any, $n = p^2$.

(c) If $n = pq$, where $p > 3$ and $q > 3$ are distinct primes, prove that $p \equiv q \equiv 2 \pmod 3$. [8]

Question 6

(i) Determine whether or not the quadratic congruence

$$3x^2 + 6x - 2 \equiv 0 \pmod{47}$$

has solutions. [4]

(ii) Evaluate the Legendre symbol $(-39/67)$. [3]

(iii) Use the Law of Quadratic Reciprocity (LQR) to prove that for an odd prime $p \neq 3$,

$$(3/p) = \begin{cases} 1, & \text{if } p \equiv \pm 1 \pmod{12}, \\ -1, & \text{if } p \equiv \pm 5 \pmod{12}. \end{cases}$$ [4]

Question 7

(i) Determine a simple finite continued fraction for $57/32$ and hence find the solution of the linear Diophantine equation

$$57x - 32y = 1$$

in which y takes its least positive value. [5]

(ii) Determine the irrational number α whose continued fraction is $[0, 1, \langle 1, 2 \rangle]$. Write down the convergents C_1, C_2, C_3, C_4, C_5 and C_6 of α and indicate (with brief justification) which is the first convergent accurate to within 0.01 as an approximation to α. [6]

Question 8

(i) Given that the continued fraction of $\sqrt{11}$ is $[3, \langle 3, 6 \rangle]$, determine two positive solutions of the Diophantine equation

$$x^2 - 11y^2 = 1.$$ [4]

(ii) Determine two Pythagorean triples, one which is primitive and one which is not, in which one of the sides is 20. [3]

(iii) Use the method of infinite descent to prove that the Diophantine equation

$$x^3 - 4y^3 = 2z^3$$

has no solution in positive integers. [4]