Answer as many questions as you wish. Full marks may be obtained by **complete** *answers to* **NINE** *questions, provided that no more than* **SIX** *questions have been selected from any one part. All questions carry equal marks.*

PART I NUMBER THEORY

Question 1

(i) Prove by induction that the formula

$$1 + 3 + 6 + 10 + \cdots + \frac{1}{2}n(n+1) = \frac{1}{6}n(n+1)(n+2)$$

holds for all integers $n \geq 1$. *[4]*

(ii) Use the Euclidean Algorithm to find the greatest common divisor of 157 and 117. Hence find integers x and y such that

$$\gcd(157, 117) = 157x + 117y.$$ *[4]*

(iii) Suppose that m and n are positive integers such that n has remainder 7 when divided by 8, and $m = 9n + 4$. Prove that $\gcd(m, n) = 1$. *[3]*

Question 2

(i) Prove that there are infinitely many primes of the form $4k + 3$. *[7]*

(ii) Prove that for any prime p it is not possible for both $8p - 1$ and $8p + 1$ to be prime. *[4]*

Question 3

(i) For each of the following statements decide whether or not it is generally true. If you believe it to be true, prove it; otherwise give a counter-example. Throughout a, b, k and n are positive integers.

(a) If $a \equiv b \pmod{n}$ then $k(a+1) \equiv k(b+1) \pmod{n}$;

(b) if $ka \equiv kb \pmod{n}$ then $a \equiv b \pmod{n}$;

(c) if $a^2 \equiv b^2 \pmod{n}$ then either $a \equiv b \pmod{n}$ or $a \equiv -b \pmod{n}$. *[6]*

(ii) Find the least positive integer x which simultaneously satisfies all three of the following linear congruences:

$$x \equiv 1 \pmod{5}; \quad 3x \equiv 2 \pmod{7}; \quad 6x \equiv 2 \pmod{11}.$$ *[5]*

Question 4

(i) Use Fermat's Little Theorem in showing the following.

(a) $3^{50} + 5^{50}$ is divisible by 17.

(b) If p is an odd prime then each of the numbers $(p-1)2^{p-1} + 1$ and $(p-2)2^{p-2} + 1$ is divisible by p. *[7]*

(ii) Use Wilson's Theorem in showing that 13 is the smallest prime which divides $11! - 1$. *[4]*