Question 5

The respective parts of this question are concerned with Euler's phi-function, ϕ, and the sum of the positive divisors function, σ. You may assume that each of these functions is multiplicative.

(i) Prove the formula

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$ [5]

(ii) A positive integer n has the property that $n + 3\sigma(n)$ is divisible by 4. Prove that:

(a) if $n = p^2$ (for some prime p) then $p \equiv 3 \pmod 4$;

(b) if $n = p^2 q$ (for distinct primes p and q) then $p = 2$. [6]

Question 6

(i) Determine whether or not the quadratic congruence

$$2x^2 + 7x + 1 \equiv 0 \pmod{17}$$

has solutions. [4]

(ii) Evaluate the Legendre symbol $(-47/73)$. [3]

(iii) Use Gauss's Lemma to show that 2 is a quadratic residue of any prime of the form $8k + 7$. [4]

Question 7

(i) The number x has continued fraction

$$x = [1; 1, 2, 2, 3, 3, 4, 4, \ldots].$$

Write down the convergents C_0, C_1, C_2, C_3, C_4, C_5 and C_6 of x. Estimate the accuracy of C_5 by obtaining an upper bound for $|x - C_5|$. [5]

(ii) Determine the irrational number which has periodic continued fraction $[0; 3, \overline{1, 2}]$. [6]

Question 8

(i) Given that the continued fraction of $\sqrt{7}$ is $[2; \overline{1, 1, 1, 4}]$, determine two solutions of Pell's equation

$$x^2 - 7y^2 = 1.$$ [4]

(ii) Of the numbers 240, 245 and 270, one can be written as a sum of two squares whilst the other two cannot. Identify which two cannot be written as the sum of two squares, justifying your answer, and express the third as a sum of two squares. [4]

(iii) Use the method of infinite descent to show that the equation

$$x^2 = 2y^2$$

has no solutions in positive integers. [3]