Question 10

(i) Let h be the function

$$\mathrm{Pr}[s, \mathrm{Cn}[\exp, \mathrm{Cn}[s, \mathrm{id}_2^3], \mathrm{Cn}[s, \mathrm{id}_3^3]]]$$

where exp is the function defined by $\exp(x, y) = x^y$.

Compute the values of

(a) $h(3, 0)$,

(b) $h(3, 2)$. [4]

(ii) In this part you may present your arguments using either formal or informal definitions.

(a) Show that the function exp, as in part (i), is primitive recursive by defining it in terms of the sum function and the initial functions. [3]

(b) Show that the function f of 3 arguments defined by

$$f(x_1, x_2, x_3) = x_2^{(x_1^{x_3})}$$

is primitive recursive by defining it in terms of exp, the initial functions and, if you wish, the sum and product functions. [2]

(iii) Write down the pairs (x_1, x_2) of natural numbers for which $\mathrm{Mn}[f](x_1, x_2)$ is defined, where f is the function defined by

$$f(x_1, x_2, y) = x_2^{y+1} \cdot ((x_1 + y) \dot{-} x_2)$$ [2]

Question 11

(i) A Turing machine has a configuration with left number 53 and right number 82. Find the left and right numbers of the configuration which results when the scanning head is moved one square to the left, and draw the resulting configuration. [3]

(ii) In parts (a), (b) and (c) below, you may use any of the recursive functions, or results about them, given in the Logic Handbook without proving that they are recursive. You may also give your answers as formal or informal definitions.

(a) Show that the condition "$x > y$" on pairs (x, y) of natural numbers is primitive recursive. [3]

(b) Show that the function Max defined by

$$\mathrm{Max}(x, y) = \begin{cases} x, & \text{if } x \geq y, \\ y, & \text{if } x < y, \end{cases}$$

is primitive recursive. [$1\frac{1}{2}$]

(c) Show that the function f defined by

$$f(x, y) = \begin{cases} x^3 y, & \text{if } 21 \leq \mathrm{Max}(x, 2y), \\ x + y, & \text{if } 3x + 4y = 41, \\ 3, & \text{otherwise}, \end{cases}$$

is primitive recursive. [$3\frac{1}{2}$]