Question 6

(i) Decide whether or not the quadratic congruence

$$2x^2 + 7x + 4 \equiv 0 \pmod{29}$$

has solutions. [4]

(ii) Evaluate the Legendre symbol $(89/43)$. [3]

(iii) Prove that for an odd prime $p \neq 3$,

$$(3/p) = \begin{cases} 1, & \text{if } p \equiv 1, 11 \pmod{12}, \\ -1, & \text{if } p \equiv 5, 7 \pmod{12}. \end{cases}$$ [4]

Question 7

(i) Write down the convergents $C_0, C_1, C_2, C_3, C_4, C_5$ and C_6 for the continued fraction $x = [1; 1, 2, 2, 3, 3, 4, 4, \ldots]$. Estimate the accuracy of C_5 by obtaining an upper bound for $|x - C_5|$. (There is no need to simplify your answer.) [6]

(ii) Determine the irrational number which has periodic continued fraction $[1; 1, \overline{2, 4}]$. [5]

Question 8

(i) For each of the numbers 360 and 364 decide whether or not it can be expressed as the sum of the squares of two positive integers. If it can be so expressed, determine two suitable squares to write it in this way; if not, explain why not. [5]

(ii) Use the method of descent to show that the Diophantine equation $x^3 + 9y^3 = 3z^3$ has no solutions in positive integers. [6]