Question 1

(i) Prove by induction the formula

$$1 + 5 + 12 + 22 + \cdots + \tfrac{1}{2}n(3n-1) = \tfrac{1}{2}n^2(n+1).$$ [4]

(ii) Use the Euclidean Algorithm to find integers x and y such that $\gcd(17, 143) = 17x + 143y$. Write down the general solution of this equation and the particular solution in which x takes its least positive value. [4]

(iii) A sequence $\{a_n\}$ of integers is defined recursively by

$$a_1 = a_2 = 1, \quad a_n = a_{n-1} + 2a_{n-2}, \text{ for } n \geq 3.$$

Prove that, for all $n \geq 1$, $\gcd(a_n, a_{n+1}) = 1$. [3]

Question 2

(i) Suppose that there are only finitely many distinct primes, say $p_1, p_2, \cdots, p_n$. Show that the number $p_1 + p_2 p_3 \cdots p_n$ is not divisible by any prime in this list and hence deduce that there must be infinitely many primes. [4]

(ii) Show that if p and $p+2$ are both primes, where $p > 3$, then their sum is divisible by 12. [3]

(iii) Prove that if a, $a+d$ and $a+2d$ are primes, where a and d are integers with $a > 3$ and $d \geq 1$, then d is a multiple of 6. [4]

Question 3

(i) Show that the final (units) digit of the fourth power, n^4, of any integer n can be one of just four values. Hence, by considering the final digit of $x^4 + y^4 + z^4$, show that if $n \equiv 4 \pmod 5$ then n cannot be written as a sum of three fourth powers of integers. [5]

(ii) Find the least positive integer x which simultaneously satisfies all three of the following linear congruences:

$$2x \equiv 3 \pmod 5; \quad 3x \equiv 4 \pmod 7; \quad 4x \equiv 9 \pmod{11}.$$ [6]

Question 4

(i) Use Fermat's Little Theorem in showing the following.

(a) $2^{100} + 3^{100}$ is divisible by 97. [3]

(b) $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$, where p and q are distinct primes. [3]

(ii) Prove Wilson's Theorem: If p is prime then $(p-1)! \equiv -1 \pmod p$. [5]

Question 5

This question is concerned with the multiplicative functions τ, σ and ϕ.

(i) Prove that $\tau(n)$ is odd if and only if n is a square. [3]

(ii) An integer n is said to be *deficient* if $\sigma(n) < 2n$. Prove that $n = pq$, where p and q are distinct primes, is deficient with the exception of one value of n. [4]

(iii) Let $n = 2(2p-1)$, where p and $2p-1$ are odd primes. Show that $\phi(n) = \phi(n+2)$. [4]