Question 13

(i) Show that the following formula takes truth value 1 under all interpretations of its symbols.

$$((\exists x\, x = y \,\&\, \neg(\forall x(x = y \longrightarrow \forall x\, x = y) \longrightarrow \forall x\, x = y)) \longrightarrow (\neg\exists x\, x = y \vee \neg\forall x\, x = y))$$

[3]

(ii) The following is a correct (but contorted) proof from which the assumption numbers have been omitted.

(1)	$(\neg\phi \longleftrightarrow \psi)$	Ass	1
(2)	$\forall x(\neg\psi \,\&\, \theta)$	Ass	2
(3)	$(\neg\psi \,\&\, \theta)$	UE, (2)	2
(4)	$(\phi \longleftrightarrow \neg\psi)$	Taut, (1)	1
(5)	$\exists x(\neg\phi \longleftrightarrow \psi)$	Ass	5
(6)	$(\theta \,\&\, \phi)$	Taut, (3), (4)	1,2
(7)	$\exists x(\theta \,\&\, \phi)$	EI, (6)	1,2
(8)	$\exists x(\theta \,\&\, \phi)$	EH, (7)	2,5
(9)	$(\forall x(\neg\psi \,\&\, \theta) \longrightarrow \exists x(\theta \,\&\, \phi))$	CP, (8)	5

(a) Write down the assumptions in force on each line. [$2\frac{1}{2}$]

(b) Write down the tautology used on line (6). [$\frac{1}{2}$]

(c) For each of the following possible line (10)s, write down whether or not the proof would still be correct were the line to be added.

(A) 5 (10) $(\phi \longleftrightarrow \neg\psi)$ EH, (4) NO
(B) 1 (10) $\forall x(\phi \longleftrightarrow \neg\psi)$ UI, (4) NO

Answer YES or NO in each case. [2]

(iii) The rule UE states that:

> if the formula $\forall v\phi$ occurs on a certain line in a formal proof and
> τ is a term which may be freely substituted for a variable v in ϕ,
> then on any subsequent line we may derive the formula $\phi(\tau/v)$,
> and this formula will depend on the same assumptions as did $\forall v\phi$.

Give a suitable example from everyday mathematics to show that the rule is no longer valid if the underlined passage, which gives a condition on the term τ, is omitted. [3]

Question 14

(i) Which of the following terms are freely substitutable for x in the formula

$$\forall z(\exists y(x' + t) = y \,\&\, \forall x \forall t(t \cdot y) = x).$$

(a) $(t + x)$ ✓
(b) $(0' + z)$
(c) $(0' + x)$

Answer YES or NO in each case. [2]

(ii) Let ϕ and ψ be formulas, where the variable x does not appear free in ψ but might appear free in ϕ. Give formal proofs to show each of the following.

(a) $\neg\exists x\phi, \phi \vdash (\psi \,\&\, \neg\psi)$ [$2\frac{1}{2}$]

(b) $\neg\exists x\phi \vdash \forall x\, \neg\phi$ [$2\frac{1}{2}$]

(c) $\forall x\, \neg\phi \vdash \neg\exists x\phi$ [4]

[For each of (b) and (c) you may use the result of (a), if you wish.]