Question 9

(i) (a)

Next Instruction	R_1	R_2	R_3
1	2	1	3
2	2	1	3
3	2	1	3
4	2	2	3
5	2	2	4
1	2	2	4
6	2	2	4
STOP	2	2	4

(i) (b) $f_P^1: \mathbb{N} \to \mathbb{N}$ is defined by $f_P^1(n) = n$. It is a total function.

(i) (c)

$f_P^2(n, m)$ is a total function. If $m \leq n$ then the loop terminates when the contents of registers 1 and 2 are equal. If $m > n$ then the loop terminates when the contents of registers 1 and 3 are equal.

[[$f_P^2: \mathbb{N}^2 \to \mathbb{N}$ is defined by $f_P^2(n, m) = \begin{cases} n+m, & \text{if } m > n \\ n, & \text{otherwise} \end{cases}$.]]

$f_P^3: \mathbb{N}^3 \to \mathbb{N}$ is defined by $f_P^3(n, m, p)$ is not a total function. If n is less than both m and p then the contents of register 1 is never equal to the contents of registers 2 or 3. Therefore the loop never terminates.

(ii)

1 J(1, 2, 7)
2 S(2)
3 J(1, 2, 6)
4 S(3)
5 J(1, 1, 2)
6 C(3, 1).

Question 10

(i)

The characteristic function for the relation >,

$\chi_>: \mathbb{N}^2 \to \mathbb{N}$ be defined by $\chi_>(n, m) = sg\,(n \dot{-} m)$.

If $n > m$ then $n \dot{-} m > 0$ and so $\chi_>(n, m) = 1$.
If $n \leq m$ then $n \dot{-} m = 0$ and so $\chi_>(n, m) = 0$.

Since $\chi_>$ is obtained by substitution from the total primitive recursive functions sg and $\dot{-}$ then $\chi_>$ is also a total primitive recursive function.

Therefore $>$ is a primitive recursive relation.

(ii)

Since A and B are primitive recursive sets then their characteristic functions χ_A and χ_B are primitive recursive.

Let $\chi_{A\cap B}(n) =$ mult $(\chi_A(n), \chi_B(n))$. As $\chi_{A\cap B}(n)$ is obtained by substitution from the primitive recursive functions mult, χ_A, χ_B then $\chi_{A\cap B}$ is a primitive recursive function.

Since $\chi_{A\cap B}(n)$ equals 1 if and only if both $\chi_A(n)$ and $\chi_B(n)$ are 1 then $\chi_{A\cap B}$ is the characteristic function for $A\cap B$. Therefore the set $A\cap B$ is primitive recursive.

(iii) Use of Unit 2 Theorem 1.5

Define the functions

$g_1(n, m) = m^5 = \exp(m, 5)$
$g_2(n, m) = 9 = C_9^2\ (n, m)$,
$g_2(n, m) = m + n = add(m, n)$,

and the relations

$R_1(n, m) \Leftrightarrow X_E(nm + 5)$,
$R_2(n, m) \Leftrightarrow 3n + 2m = 9000$,
$R_3(n, m) \Leftrightarrow$ not $R_1(n, m)$ and not $R_2(n, m)$.

Then we can write

$$f(n, m) = \begin{cases} g_1(n,m) & \text{if } R_1(n,m) \\ g_2(n,m) & \text{if } R_2(n,m) \\ g_3(n,m) & \text{if } R_3(n,m) \end{cases}$$

As g_1, g_2, and g_3 can be written the primitive recursive functions C_9^2, add, and exp using constants then g_1, g_2, and g_3 are primitive recursive functions.

The characteristic function of the relation R_1, $\chi_{R1}(n,m) = \chi_E(mn+5)$. As χ_{R_1} is obtained by substitution from the primitive recursive functions χ_E, mult and add using constants, then it is a primitive recursive function. Hence R_1 is a primitive recursive relation.

The characteristic function of the relation R_2, $\chi_{R2}(n,m) = \chi_{eq}(3n+2m, 9000)$. As χ_{R_2} is obtained by substitution from the primitive recursive functions χ_{eq}, mult and add using constants, then it is a primitive recursive function. Hence R_2 is a primitive recursive relation.

Using the result of Unit 2, Problem 1.10, then R_3 is also a primitive recursive relation.

From the definition of R_3 it follows that the set of relations R_1, R_2, and R_3 are exhaustive.

If the relation R_1 holds then both n and m are odd. If the relation R_2 holds then n is even. Therefore R_1 and R_2 are mutually exclusive. From the definition of R_3, if the relation R_3 holds then neither R_1 or R_2 holds. Therefore R_1, R_2 and R_3 are mutually exclusive.

Since all the conditions required for the use of Theorem 1.5 of Unit 2 hold then it follows that f is primitive recursive.

Question 11

(i)(a)

$g(n_1, n_2, n_3) = f(U_3^3(n_1, n_2, n_3), U_1^3(n_1, n_2, n_3), h(n_1, n_2, n_3))$,

where $h(n_1, n_2, n_3) = \text{zero}(U_3^3(n_1, n_2, n_3))$

As h is obtained by substitution from the basic primitive functions zero and U_3^3 then h is primitive recursive.

As g is obtained by substitution from the primitive recursive functions f, U_1^3, U_3^3, and h then g is a primitive recursive function.

(i)(b)

Let $\exp(n, 0) = f(n)$, where $f(n) = \text{succ}(\text{zero}(n))$.
and $\exp(n, m + 1) = g(n, m, \exp(n, m))$
where $g(n_1, n_2, n_3) = \text{mult}(U_1^3(n_1, n_2, n_3), U_3^3(n_1, n_2, n_3))$.

As f is obtained by substitution from the primitive recursive functions succ and zero then it is primitive recursive.

g is a primitive recursive function since it is obtained by substitution from mult and the basic primitive recursive functions U_1^3, and U_3^3. As exp is formed by primitive recursion from the primitive recursive functions f and g, then exp is a primitive recursive function.

(ii)

Consider the relation T given by $T(n, y) \Leftrightarrow n < 3^y$.
The relation < is primitive recursive [HB p21] and the function exp(3, y) is primitive recursive by part (i)(a).

$\chi_T(n, y) = \chi_<(n, \exp(3, y))$, is obtained by substitution from the primitive recursive functions $\chi_<$ and exp using constants. Hence it is primitive recursive [HB p21 result of problem 1.10].

By Theorem 3.5 [HB p23] the function $g : \mathbb{N}^2 \to \mathbb{N}$ given by

$g(n, z) = \mu y \le z\ T(n, y)$

is primitive recursive.

As $n < 3^n$ then a suitable bound on y in terms of n is n, so

$$f(n) = \mu y \le n\ \ T(n, y) = g(n, n).$$

As f is obtained from the primitive recursive function g by substitution then f is primitive recursive [Unit 2 Problem 1.4].

Question 12

(i) [[Similar to Unit 3, Problem 3.3.]]

For any natural number n, define a URM program by

(1) S(1)
(2) C(1, 1)
⋮
(n + 1) C(n, n)

This implements the identity function successor function succ since it halts with the original value + 1 in register 1. As a program exists for each $n \in \mathbb{N}$, where $n > 0$, then there are infinitely many such programs.

(ii) [[See Unit 3, Problem 3.4.]]

We need to store the contents of register 1 in a register not used by the program P, run P, restore the contents of register 1, and add 1 to it. The program P^* can be created by concatenating the programs

(1) C(1, ρ(P) + 1),
P, and
(1) C(ρ(P) + 1,1)
(2) S(1).

If f_P^1 is total, then P* saves the input in a register not used by P, and then executes P. As f_P^1 is total, the last two instructions of P* are also executed. These add one to the original value of register 1. Therefore P* computes the successor function..

If f_P^1 is not total, then P will not halt for some input n. For this input, P* will execute the first instruction. As this does not affect register 1 the program P will not halt.

So P* computes the successor function precisely when the function f_P^1 is total.

(iii)(a) [[See Unit 3, Problem 3.4.]]

To test if a number $e \in$ *Tot* check if e codes a URM program. If it does not then $e \notin$ Tot. If it does code a URM program then the instructions of the program P can be recovered from e. Then P^* which computes the successor function can then be created as described in part (ii).

The code number e^* of P^* can then be determined. If the set X is recursive then there is an algorithm for deciding if a number $e^* \in X$. As $e^* \in X$ if and only if $e \in$ *Tot* then we can determine whether $e \in$ *Tot*.

(iii)(b) Theorem 3.2 of Unit 3 states that there is no algorithm which determines whether $e \in$ Tot. As we have found one then the assumption that the set X is recursive must be false.

Question 13

(i)

Let φ be the subformula $\exists y\, y' = x$; ψ be $\neg y' = x$; and χ be $\exists y\,(y' = x \leftrightarrow \exists y\, y' = x)$.

The given formula can then be written as $((\neg\varphi \to (\psi \,\&\, \chi)) \to (\psi \vee \varphi))$

A truth table for this formula is

φ	ψ	χ	$((\neg\varphi \to (\psi \,\&\, \chi)) \to (\psi \vee \varphi))$
1	1	1	0 1 1 1 1 1 1 1 1 1
1	1	0	0 1 1 1 0 0 1 1 1 1
1	0	1	0 1 1 0 0 1 1 0 1 1
1	0	0	0 1 1 0 0 0 1 0 1 1
0	1	1	1 0 1 1 1 1 1 1 1 0
0	1	0	1 0 0 1 0 0 1 1 1 0
0	0	1	1 0 0 0 0 1 1 0 0 0
0	0	0	1 0 0 0 0 0 1 0 0 0
			(1) (2) (1) (3) (1)

Since column 3 is all ones then the formula takes the truth value 1 under all interpretations.

(ii)(a)

Line	1	2	3	4	5	6	7	8	9
Ass.	1	1	3	4	1,3	1	1,3	1,4	1

(ii)(b)

$(((\phi \,\&\, \neg\psi) \,\&\, (\psi \to \theta)) \to (\psi \to \theta))$.

(ii)(c)

(A) YES (B) NO. [[ψ or θ may contain free occurrence of x]]

(iii)

As y cannot be freely substituted for x in $\exists y\, x' = y$ then the proof is not valid.

Take the standard interpretation $\mathcal{N}$ with domain $\mathbb{N}$. In this interpretation $\forall x\, \exists y\, x' = y$ is true as there is always a number y which is equal to the successor of x.

There is not a number y which is equal to its successor so $\exists y\, y' = y$ is false. Therefore $\exists y\, y' = y$ is not a logical consequence of $\forall x\, \exists y\, x' = y$.

Question 14

(i)

(a) NO [[z becomes bound]] (b) NO [[t becomes bound]] (c) YES

(ii)(a)

1	(1)	$\forall x\, \forall y\, (x + y') = z$	Ass
1	(2)	$\forall y\, (x' + y') = z$	UE, 1
1	(3)	$(x' + y') = z$	UE, 2
1	(4)	$\exists y\, (x' + y) = z$	EI, 3
1	(5)	$\exists x\, \exists y\, (x' + y) = z$	EI, 4

Therefore $\forall x\, \forall y\, (x + y') = z \vdash \exists x\, \exists y\, (x' + y) = z$.

(ii)(b)

1	(1)	$(\phi \,\&\, \forall x\, \neg\psi)$	Ass
2	(2)	$\exists x\, (\neg\phi \vee \theta)$	Ass
3	(3)	$(\neg\phi \vee \theta)$	Ass
4	(4)	$\forall x\, (\theta \rightarrow \psi)$	Ass. Contradiction
1	(5)	$\forall x\, \neg\psi$	Taut, 1
1	(6)	$\neg\psi$	UE, 5
4	(7)	$(\theta \rightarrow \psi)$	UE, 4
1, 4	(8)	$\neg\theta$	Taut, 6, 7
1, 3, 4	(9)	$\neg\phi$	Taut, 3, 8
1	(10)	ϕ	Taut, 1
1, 3, 4	(11)	$(\phi \,\&\, \neg\phi)$	Taut, 9, 10
1, 2, 4	(12)	$(\phi \,\&\, \neg\phi)$	EH, 11
1, 2	(13)	$(\forall x(\theta \rightarrow \psi) \rightarrow (\phi \,\&\, \neg\phi))$	CP, 12
1, 2	(14)	$\neg\, \forall x(\theta \rightarrow \psi)$	Taut, 13
1	(15)	$(\exists x\, (\neg\phi \vee \theta) \rightarrow \neg\, \forall x(\theta \rightarrow \psi))$	CP, 14

The assumption that x does not occur free in ϕ is required for the use of EH on line (11).

Question 15

(i) [[Looks as if both sides equal (**0** . x).]]

-	(1)	$((\mathbf{0} . x) + (x . \mathbf{0})) = ((\mathbf{0} . x) + (x . \mathbf{0}))$	II
2	(2)	$\forall x\ (x . \mathbf{0}) = \mathbf{0}$	Ass. Q6
2	(3)	$(x . \mathbf{0}) = \mathbf{0}$	UE, 2
2	(4)	$((\mathbf{0} . x) + (x . \mathbf{0})) = ((\mathbf{0} . x) + \mathbf{0})$	Sub, 1, 3
5	(5)	$\forall x\ (x + \mathbf{0}) = x$	Ass. Q4
5	(6)	$((\mathbf{0} . x) + \mathbf{0}) = (\mathbf{0} . x)$	UE, 5
2, 5	(7)	$((\mathbf{0} . x) + (x . \mathbf{0})) = (\mathbf{0} . x)$	Sub, 4, 6
-	(8)	$((\mathbf{0} + \mathbf{0}) . x) = ((\mathbf{0} + \mathbf{0}) . x)$	II
5	(9)	$(\mathbf{0} + \mathbf{0}) = \mathbf{0}$	UE, 5
5	(10)	$(\mathbf{0} . x) = ((\mathbf{0} + \mathbf{0}) . x)$	Sub, 8, 9
2, 5	(11)	$((\mathbf{0} . x) + (x . \mathbf{0})) = ((\mathbf{0} + \mathbf{0}) . x)$	Sub, 7, 10
2, 5	(12)	$\forall x\ ((\mathbf{0} . x) + (x . \mathbf{0})) = ((\mathbf{0} + \mathbf{0}) . x)$	UI, 11

As the assumptions are axioms of Q then the sentence is a theorem of Q.

(ii) In the interpretation $\mathcal{N}^{**}$ let x = α. Then $x' = \alpha$. There is no value y such that $(y + \alpha) = \alpha$. Therefore in $\mathcal{N}^{**}$ the sentence $\exists y \forall x\ (y + x') = x'$ is not true.

All the axioms of Q hold in $\mathcal{N}^{**}$. As $\exists y\ \forall x\ (y + x') = x'$ does not hold in $\mathcal{N}^{**}$ then, it follows by the Correctness Theorem, the sentence is not a theorem of Q.

(iii)

1	(1)	$\forall x\ (x + \mathbf{0}) = x$	Ass. Q4
1	(2)	$(y' + \mathbf{0}) = y'$	UE, 1
1	(3)	$\forall y\ (y' + \mathbf{0}) = y'$	UI, 2
1	(4)	$\exists x\ \forall y\ (y' + x) = y'$	EI, 3

As the assumption is an axiom of Q then the sentence is a theorem of Q.

Question 16

(i) Solution by Linda Brown.

Suppose that theory T is not consistent but has an interpretation
Hence there is a sentence of T, Φ say, such that $\vdash_T \Phi$ and $\vdash_T \neg \Phi$,
i.e. both Φ and $\neg \Phi$ are theorems of T

By the Correctness Theorem both Φ and $\neg \Phi$ are true in every interpretation in which the sentences of T are true and so must be true in the interpretation of T

However a sentence cannot be both true and false in the same interpretation and this contradicts our original supposition

Hence a theory which has an interpretation is consistent

(ii)(a)

The standard interpretation $\mathcal{N}$, is an interpretation for each of the theories Q and CA. Therefore by part (i) both Q and CA are consistent.

(ii)(b)

Q is not complete. $\forall x\ \forall y\ (x + y) = (y + x)$ is a sentence in Q. This sentence is true in the interpretation $\mathcal{N}$ of Q but false in the interpretation $\mathcal{N}^{**}$ of Q.

As theory CA consists of all the sentences of the formal language that are true in the standard interpretation $\mathcal{N}$, i.e. for every sentence either $\vdash_{CA} \Phi$ or $\vdash_{CA} \neg \Phi$, therefore CA is complete. (Linda Brown)

(iii)

If there is an algorithm for deciding which sentences are theorems of CA then CA must be recursively axiomatizable. By Unit 8, Theorem 2.4, CA is not recursively axiomatizable. Hence there is no such algorithm.

END OF PART II SOLUTIONS