

Question 9**(i)(a) (2 marks)**

Next Instruction	R ₁	R ₂	R ₃	R ₄
1	2	3	1	0
2	2	3	1	0
3	2	4	1	0
4	2	4	1	1
1	2	4	1	1
5	2	4	1	1
STOP	4	4	1	1

(i)(b) (5 marks)

$f_p^1: \mathbb{N} \rightarrow \mathbb{N}$ is defined by $f_p^1(n) = 0$. It is a total function.

$f_p^2: \mathbb{N}^2 \rightarrow \mathbb{N}$ is defined by $f_p^2(n, m) = m$. It is a total function.

$f_p^3: \mathbb{N}^3 \rightarrow \mathbb{N}$ is defined by $f_p^3(n, m, p) = m + p$. It is a total function.

$f_p^4: \mathbb{N}^4 \rightarrow \mathbb{N}$ is defined by $f_p^4(n, m, p, q) = \begin{cases} m + p - q, & \text{if } p \geq q \\ \text{undefined,} & \text{otherwise.} \end{cases}$

(ii) (4 marks)

- 1 S(2)
- 2 S(2)
- 3 J(1, 2, 6)
- 4 Z(1)
- 5 S(1).

Question 10**(i) (2½ marks)** See Unit 2, Problem 1.9Let $\chi_{\leq}: \mathbb{N}^2 \rightarrow \mathbb{N}$ be defined by $\chi_{\leq}(n, m) = \overline{\text{sg}}(n \dot{-} m)$.If $n > m$ then $n \dot{-} m > 0$ and so $\chi_{\leq}(n, m) = 0$.If $n \leq m$ then $n \dot{-} m = 0$ and so $\chi_{\leq}(n, m) = 1$.Since χ_{\leq} is obtained by substitution from the total primitive recursive functions $\overline{\text{sg}}$ and $\dot{-}$ then χ_{\leq} is also a total primitive recursive function.Therefore χ_{\leq} is a characteristic function for the relation \leq and so it is primitive recursive.**(ii) 2½ marks** See Unit 2, Problem 1.10The characteristic function χ_R of the relation R is primitive recursive.Let $\chi_T(n_1, n_2, \dots, n_k) = \overline{\text{sg}}(\chi_R(n_1, n_2, \dots, n_k))$.Therefore the relation T is primitive recursive since its characteristic function is obtained by substitution from the primitive functions $\overline{\text{sg}}$ and χ_R .**(iii) 6 marks** Use of Unit 2 Theorem 1.5

Define the functions

$$g_1(m, n) = 7 = C_7^2(m, n),$$

$$g_2(m, n) = mn = \text{mult}(m, n),$$

$$g_3(m, n) = n^3 = \text{exp}(n, 3)$$

and the relations

$$R_1(n, m) \Leftrightarrow 60 \leq \min(n, m),$$

$$R_2(n, m) \Leftrightarrow 4n + 3m = 400,$$

$$R_3(n, m) \Leftrightarrow \text{not } R_1(n, m) \text{ and not } R_2(n, m).$$

Then we can write

$$f(n, m) = \begin{cases} g_1(m, n) & \text{if } R_1(m, n) \\ g_2(m, n) & \text{if } R_2(m, n) \\ g_3(m, n) & \text{if } R_3(m, n) \end{cases}$$

As g_1 , g_2 , and g_3 can be written the primitive recursive functions C_7^2 , mult , and exp using constants then g_1 , g_2 , and g_3 are primitive recursive functions.

The characteristic function of the relation R_1 , $\chi_{R_1}(n, m) = \chi_{\leq}(60, \min(n, m))$. As χ_{R_1} is obtained by substitution from the primitive recursive functions χ_{\leq} , and \min using constants, then it is a primitive recursive function. Hence R_1 is a primitive recursive relation.

The characteristic function of the relation R_2 , $\chi_{R_2}(n, m) = \chi_{\text{eq}}(4n + 3m, 400)$. As χ_{R_2} is obtained by substitution from the primitive recursive functions χ_{eq} , mult and add using constants, then it is a primitive recursive function. Hence R_2 is a primitive recursive relation.

Using the result of Problem 1.10, then R_3 is also a primitive recursive relation.

From the definition of R_3 it follows that the set of relations R_1 , R_2 , and R_3 are exhaustive.

If the relation R_1 holds then $m \geq 60$ and $n \geq 60$. Therefore $4n + 3m \geq 4 * 60 + 3 * 60 = 420$. Since R_2 does not hold then R_1 and R_2 are mutually exclusive. From the definition of R_3 , if the relation R_3 holds then neither R_1 or R_2 holds. Therefore R_1 , R_2 and R_3 are mutually exclusive.

Since all the conditions required for the use of Theorem 1.5 of Unit 2 hold then it follows that f is primitive recursive.

Question 11

(i)(a) (2 marks)

Let $C_2^3(n_1, n_2, n_3) = \text{succ}(\text{succ}(\text{zero}(U_1^3(n_1, n_2, n_3))))$.

As C_2^3 is obtained by substitution from the basic primitive recursive functions succ , zero , and U_1^3 then C_2^3 is a primitive recursive function.

(i)(b) (3 marks)

Let $\text{mult}(n, 0) = \text{zero}(n)$,
and $\text{mult}(n, m + 1) = g(n, m, \text{mult}(n, m))$
where $g(n_1, n_2, n_3) = \text{add}(U_1^3(n_1, n_2, n_3), U_3^3(n_1, n_2, n_3))$.

g is a primitive recursive function since it is obtained by substitution from add and the basic primitive recursive functions U_1^3 , and U_3^3 . Since zero is a basic primitive recursive functions and g is a primitive recursive function then mult is a primitive recursive function.

(ii)(a) (2 marks)

Let R be the relation 'the remainder of y on division by 4 is 1'. A characteristic function for R is,
 $\chi_R(y) = \chi_{\text{eq}}(\text{rem}(y, 4), 1)$.

As χ_R is obtained by substitution from the primitive recursive functions χ_{eq} and rem using constants then χ_R is a primitive recursive function. Therefore R is a primitive recursive relation.

(ii)(b) 4 marks

An improved version by Lisette Petrie.

$f(n) = \mu y (n < y \text{ and the remainder of } y \text{ on division by 4 is 1})$

Consider the relation T given by

$T(n, y) \Leftrightarrow n < y \text{ and the remainder of } y \text{ on division by 4 is 1.}$

The relation $<$ is primitive recursive [HB p21] and the relation "the remainder of y on division by 4 is 1" is primitive recursive by part (a).

Then $\chi_T(n, y) = \chi_{<}(n, y) \text{ and } \chi_R(y)$,

which is obtained by substitution from the primitive recursive functions $\chi_{<}$ and χ_R and so is primitive recursive [HB p21 result of problem 1.10].

By Theorem 3.5 [HB p23] the function $g : \mathbb{N}^2 \rightarrow \mathbb{N}$ given by

$$g(n, z) = \mu y \leq z T(n, y)$$

is primitive recursive.

A suitable bound on y in terms of n is $n + 4$, so

$$f(n) = \mu y \leq (n + 4) T(n, y) = g(n, n + 4)$$

which is a substitution of primitive recursive functions using constants, so is primitive recursive.

Question 12

Solution by Wilson Stother.

(i) (2 marks)

For any natural number n , define the URM program $I(n)$ by

(1) $C(1, n)$.

This implements the identity function id since it halts with the original value in register 1.

Thus there are infinitely many such programs.

(ii) (5 marks)

Given a program P , first determine the maximum register number $\rho(P)$ used by P .

Now create P^* by concatenating the following programs

(1) $C(1, \rho(P)+1)$,

P , and

(1) $C(\rho(P)+1, 1)$.

If f_p^1 is total, then P^* saves the input in a register not used by P , executes P , and, as f_p^1 is total, moves on to the last instruction of P^* which restores the original value of register 1. Thus P^* computes the identity function id in this case.

If f_p^1 is not total, then P^* will not halt for some input n . For this input, P^* will execute the first instruction. This does not affect register 1, so the program P will not halt. Thus, P^* will *not* halt for this input. Hence the function computed by P^* is not total, and so P^* cannot compute the identity function.

(iii) (4 marks)

Suppose that X is recursive. Then there is an algorithm for testing whether a code number e^* is in X .

Tot is the set of code numbers of URMs which compute total functions of one variable.

We now have the following algorithm for deciding whether a given integer e is in Tot .

There is a simple algorithm to check whether e is the code number of a URM.

If not, then e is not in Tot .

Otherwise, we can recover P from e , and construct P^* as in (ii).

Let P^* have code number e^* .

From part (ii), $e \in Tot$ if and only if $e^* \in X$.

The existence of such an algorithm contradicts Theorem 3.2, so our assumption must be false.

Hence X is not recursive.

Question 13

(i) (3 marks)

Let θ be the formula $x' = y$; ϕ be $\exists x x' = y$; and ψ be $\forall x (x' = y \vee \exists x x' = y)$.

The given formula can then be written as $((\theta \vee \phi) \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\theta)$.

A truth table for this formula is

θ	ϕ	ψ	$((\theta \vee \phi) \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\theta)$
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	0
			(2) (3) (4) (2) (3) (2)

Since column 4 is all ones then the formula takes the truth value 1 under all interpretations.

(ii)(a) (2½ marks)

(a) 2 1/2 marks. Same as 2003 with lines 1 and 2 interchanged.

Line	1	2	3	4	5	6	7	8	9
Ass.	1	2	1	4	1,2	1,2	1,4	1	1,4

(ii)(b) (½ mark)

$$\frac{(\theta \rightarrow \psi), (\psi \& \phi)}{(\theta \rightarrow \psi)}$$

(ii)(c) (2 marks)

(A) NO (Correction by Linda Brown). (B) YES.

(iii) 3 marks. This is the solution given to Problem 2.8 in Unit 5.

Let ψ be the formula $v = 0$, and ϕ be the same formula $v = 0$. ϕ is trivially a logical consequence of ψ and v occurs freely in ψ .

In the standard interpretation \mathcal{N} give v the value 0 in the domain \mathbb{N} . Then the formula $v = 0$ is true, but $\forall v v = 0$ derived by use of the UI rule is false.

Question 14**(i) (2 marks)**

$$\forall z (\forall x \exists t t = (y.x) \leftrightarrow \forall y (y' + z) = (x + t))$$

(a) NO (b) NO (c) YES

(ii)(a) (3 marks)

1	(1)	$\exists y \forall x (x + y) = y$	Ass
2	(2)	$\forall x (x + y) = y$	Ass
2	(3)	$(x' + y) = y$	UE, 2
2	(4)	$\exists y (x' + y) = y$	EI, 3
2	(5)	$\exists x \exists y (x' + y) = y$	EI, 4
1	(6)	$\exists x \exists y (x' + y) = y$	EH, 5

Therefore $\exists y \forall x (x + y) = y \vdash \exists x \exists y (x' + y) = y$.**(ii)(b) (6 marks)**

1	(1)	ϕ	Ass
2	(2)	$\forall x (\theta \vee \neg \phi)$	Ass
3	(3)	$\exists x \neg (\psi \rightarrow \theta)$	Ass. Contradiction
4	(4)	$\neg (\psi \rightarrow \theta)$	Ass
4	(5)	$\neg \theta$	Taut, 4
2	(6)	$(\theta \vee \neg \phi)$	UE, 2
2,4	(7)	$\neg \phi$	Taut, 5, 6
1,2,4	(8)	$(\phi \ \& \ \neg \phi)$	Taut, 1, 7
1,2,3	(9)	$(\phi \ \& \ \neg \phi)$	EH, 8
1,2	(10)	$(\exists x \neg (\psi \rightarrow \theta) \rightarrow (\phi \ \& \ \neg \phi))$	CP, 9
1,2	(11)	$\neg \exists x \neg (\psi \rightarrow \theta)$	Taut, 10
1	(12)	$(\forall x (\theta \vee \neg \phi) \rightarrow \neg \exists x \neg (\psi \rightarrow \theta))$	CP, 11

The assumption that x does not occur free in ϕ is required for the use of EH on line (9).

Question 15 (11 marks)(i) $\theta = (\mathbf{0} + x)$.

-	(1)	$((\mathbf{0} + \mathbf{0}) + x) = ((\mathbf{0} + \mathbf{0}) + x)$	II
2	(2)	$\forall x (x + \mathbf{0}) = x$	Ass. Q4
2	(3)	$(\mathbf{0} + \mathbf{0}) = \mathbf{0}$	UE, 2
2	(4)	$((\mathbf{0} + \mathbf{0}) + x) = (\mathbf{0} + x)$	Sub, 1, 3
-	(5)	$(\mathbf{0} + (x + \mathbf{0})) = (\mathbf{0} + (x + \mathbf{0}))$	II
2	(6)	$(x + \mathbf{0}) = x$	UE, 2
2	(7)	$(\mathbf{0} + x) = (\mathbf{0} + (x + \mathbf{0}))$	Sub, 5, 6
2	(8)	$((\mathbf{0} + \mathbf{0}) + x) = (\mathbf{0} + (x + \mathbf{0}))$	Sub, 4, 7
2	(9)	$\forall x ((\mathbf{0} + \mathbf{0}) + x) = (\mathbf{0} + (x + \mathbf{0}))$	UI, 8

As the assumption is an axiom of Q then the sentence is a theorem of Q.

(ii)

In \mathcal{M}^{**} let $x = \alpha$, and $y = \alpha$. Then

$$\begin{aligned} ((x + y) + x) &= ((\alpha + \alpha) + \alpha) = (\beta + \alpha) = \beta, \text{ and} \\ (x + (y + x)) &= (\alpha + (\alpha + \alpha)) = (\alpha + \beta) = \alpha. \end{aligned}$$

All the axioms of Q hold in \mathcal{M}^{**} . As $\forall x \forall y ((x + y) + x) = (x + (y + x))$ does not hold in \mathcal{M}^{**} then, it follows by the Correctness Theorem, the sentence is not a theorem of Q.

(iii) Solution by Wilson Stother.

1	(1)	$\forall x (x + \mathbf{0}) = x$	Ass. Q4
1	(2)	$((\mathbf{0} + y) + \mathbf{0}) = (\mathbf{0} + y)$	UE, 1
-	(3)	$(\mathbf{0} + (y + \mathbf{0})) = (\mathbf{0} + (y + \mathbf{0}))$	II
1	(4)	$(y + \mathbf{0}) = y$	UE, 1
1	(5)	$(\mathbf{0} + y) = (\mathbf{0} + (y + \mathbf{0}))$	Sub, 3, 4
1	(6)	$((\mathbf{0} + y) + \mathbf{0}) = (\mathbf{0} + (y + \mathbf{0}))$	Sub, 2, 5
1	(7)	$\forall y ((\mathbf{0} + y) + \mathbf{0}) = (\mathbf{0} + (y + \mathbf{0}))$	UI, 6
1	(8)	$\exists x \forall y ((x + y) + x) = (x + (y + x))$	EI, 7

As the assumption is an axiom of Q then the sentence is a theorem of Q.

Question 16

Solution by Linda Brown.

(i) (3 marks)

Suppose that theory T is not consistent but has an interpretation

Hence there is a sentence of T , Φ say, such that $\vdash_T \Phi$ and $\vdash_T \neg \Phi$,
i.e. both Φ and $\neg \Phi$ are theorems of T

By the Correctness Theorem both Φ and $\neg \Phi$ are true in every interpretation in which the sentences of T are true and so must be true in the interpretation of T

However a sentence cannot be both true and false in the same interpretation and this contradicts our original supposition

Hence a theory which has an interpretation is consistent

(ii)(a) (3 marks)

Theory Q is consistent as a consequence of Problem 3.4
Hence $\neg 0 = 0'$ and $0 = 0'$ cannot both be theorems of Q

$\forall x \neg 0 = x'$ is an axiom of Q
Hence for $x = 0$, $\neg 0 = 0'$, i.e. $0 \neq 1$
Therefore $\neg 0 = 0'$ is a theorem of Q

Hence $0 = 0'$, i.e. $0 = 1$ cannot be a theorem of Q

(ii)(b) (1 mark)

$$\forall x (0 + x) = x$$

(iii) (4 marks)

(iii)(a) Theory CA is consistent because it has an interpretation, \mathcal{N} , by part (i)

(iii)(b) As theory CA consists of all the sentences of the formal language that are true in the standard interpretation \mathcal{N} , i.e. for every sentence either $\vdash_{CA} \Phi$ or $\vdash_{CA} \neg \Phi$, therefore CA is complete.

CA is also consistent by part (ii)(a)

CA is an extension of Theory Q because the axioms of Q are derivable in CA

Hence by Theorem 2.3, Gödel's Incompleteness Theorem, CA is not recursively axiomatizable, i.e. CA does not have a recursive set of axioms.

[Alternatively Theorem 2.4 gives this result directly]

END OF PART II SOLUTIONS