Question 29

For each of the following conics (which you may assume are non-degenerate), locate all branch points, all infinite points, and the pinch point (where they exist). Hence categorize each of them as:

(a) homeomorphic to the z-sphere because it is a single cover of it,

or

(b) a double cover of the z-sphere branched over two points but without a pinch point,

or

(c) a double cover of the z-sphere branched over two points and having a pinch point.

(i) $2z^2 + 2zw + w^2 + 2w + 1 = 0$ [3]

(ii) $z^2 - 6zw + 9w^2 + 2w + 1 = 0$ [4]

(iii) $3z^2 + 2zw + 2w + 1 = 0$ [4]

Question 30

(i) For the following covering $\widetilde{S} \longrightarrow S$, classify the surface $\widetilde{S}$ and find any branch points with their deficits.

$$S : aabcb^{-1}ded^{-1} = 1$$
$$\widetilde{S} : a_1a_2b_1c_1b_3^{-1}d_1e_1d_2^{-1} = 1; \quad a_2a_3b_2c_2b_2^{-1}d_2e_2d_1^{-1} = 1;$$
$$a_3a_1b_3c_3b_1^{-1}d_3e_3d_3^{-1} = 1.$$

[5]

(ii) Exhibit in pictorial form (or describe in some other way) each of the following:

(a) a 2-fold covering of S^2 by T^2 branched over 4 points,

(b) a 2-fold covering of T^2 by $2T^2$ branched over 2 points,

(c) a 2-fold unbranched covering of $2T^2$ by $3T^2$.

Combining the coverings of (a), (b) and (c) in an appropriate order gives a k-fold covering of S^2 by $3T^2$. Write down the value of k, and the corresponding total branch point order δ. [6]

Question 31

(i) Show that all the zeros of $F(z) = z^4 + 2z^2 + z + 5$ lie inside the annulus $\{z : 1 \leq |z| \leq 2\}$. [4]

(ii) Show that $F(z)$ has no real zeros. [2]

(iii) Show that there are no zeros of $F(z)$ on the imaginary axis. [1]

(iv) Show that there is exactly one zero of $F(z)$ inside the region bounded by $\{z = iy : 0 \leq y \leq 2\}$, $\{z = x : 0 \leq x \leq 2\}$ and $\{z : |z| = 2, \operatorname{Re} z \geq 0, \operatorname{Im} z \geq 0\}$. [3]

(v) Hence deduce that the four zeros of $F(z)$ lie one in each quadrant. [1]