undated.

Question 17

(i) Let $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be the function given by

$$f(x_1, x_2) = (x_1^2 + x_2^2,\ x_1), \quad (x_1, x_2) \in \mathbf{R}^2.$$

(a) Write down $f(\{1\} \times \mathbf{R})$. [2]

(b) Write down $f^{-1}(\{(1,0)\})$. [1]

(c) Sketch the subset

$$f^{-1}([1,4] \times [-1,1])$$

of $\mathbf{R}^2$. [3]

(ii) (a) Let $g : A \longrightarrow B$ be a function, where A and B are non-empty sets. Prove that if C_1 and C_2 are subsets of A then

$$g(C_1 - C_2) \supset g(C_1) - g(C_2).$$ [3]

(b) Give an example of a function $h : \mathbf{R} \longrightarrow \mathbf{R}$ and subsets D_1 and D_2 of $\mathbf{R}$ such that

$$h(D_1 - D_2) \neq h(D_1) - h(D_2).$$ [2]

Question 18

Let $d : \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ be defined by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ |x-1| + |y-1| + 2|x-y|, & \text{if } x \neq y. \end{cases}$$

(i) Show that d is a metric on $\mathbf{R}$. [5]

(ii) Determine the following open balls with respect to the metric d:

(a) $B_1(0)$; (b) $B_1(1)$. [3]

(iii) Show that the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$f(x) = 1 - x^2, \qquad x \in \mathbf{R},$$

is not (d,d)-continuous at 1. [3]

Question 19

Let $\mathcal{V}$ be the set of subsets of $\mathbf{R}^2$ defined as follows: $V \in \mathcal{V}$ if $(0,0) \notin V$ or $D \subset V$ where

$$D = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 = x_2\}.$$

(i) Show that $\mathcal{V}$ is a topology on $\mathbf{R}^2$. [7]

(ii) Determine the topology induced by $\mathcal{V}$ on the subset

$$\{(0,0), (0,1), (1,1)\}.$$ [4]