## Question 23

(i) Prove that if $H$ is a connected subspace of a topological space $T$ and $H \subset K \subset Cl(H)$, then $K$ is connected. [5]

(ii) For each positive integer $n$ let

$$X_n = \{(x_1, x_2) \in \mathbb{R}^2 : \ nx_2 = x_1, \ x_1^2 + x_2^2 < 1\}$$

and let $X = \bigcup_{n=1}^{\infty} X_n$.

(a) Sketch the subset $X$ of $\mathbb{R}^2$. [1]

(b) Write down the closure of $X$ in $\mathbb{R}^2$ with the usual topology. [No justification is needed.] [2]

(c) State whether or not the subspace

$$Y = X \cup \left\{(x_1, x_2) : \quad x_2 = 0, \quad \frac{1}{4} < |x_1| < \frac{3}{4}\right\}$$

of $\mathbb{R}^2$ with the usual topology is connected, and justify your answer. [3]

## Question 24

Let $d$ be the metric on the set $\mathbb{N}$ of positive integers defined by

$$d(i, j) = \left|\frac{i}{i+1} - \frac{j}{j+1}\right|, \quad i, j \in \mathbb{N}.$$

[You need *not* verify that $d$ is a metric.]
Consider the sequence $(x_n)$ where $x_n = n$ for each positive integer $n$.

(i) Prove that $(x_n)$ is a Cauchy sequence in the metric space $\{\mathbb{N}, d\}$. [4]

(ii) Show that if $k \in \mathbb{N}$ then

$$d(k, x_n) \geq \frac{1}{2k+2}$$

for each integer $n$ such that $n \geq 2k+1$. [3]

(iii) Deduce that the metric space $\{\mathbb{N}, d\}$ is not complete. [4]