Question 20

(i) Let T_1 and T_2 be topological spaces and let $\mathcal{B}$ be a basis for the topology of T_2. Prove that a function $f : T_1 \longrightarrow T_2$ is continuous if $f^{-1}(B)$ is open in T_1 for each set B in $\mathcal{B}$. [4]

(ii) Let $T_1 = \{A, \mathcal{T}\}$ where $A = \{a, b, c\}$ and

$$\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, A\}$$

and let T_2 denote $\mathbf{R}$ with the usual topology. [You need *not* verify that $\mathcal{T}$ is a topology on A.]

(a) Prove that if $g : T_1 \longrightarrow T_2$ is a function for which $g(b) = g(c)$, then g is continuous. [4]

(b) State whether or not $h : T_1 \longrightarrow T_2$ given by

$$h(a) = 0, \quad h(b) = 1, \quad h(c) = 2$$

is continuous, and justify your answer. [3]

Question 21

Let $\mathbf{N}$ be the set of positive integers and let $\mathcal{B} = (B_n)_{n \in \mathbf{N}}$ be the collection of subsets of $\mathbf{N}$ defined as follows:

$B_n = \{n\}$, if n is odd.
$B_n = \{n\} \cup \{m : m \text{ odd and } m > n\}$, if n is even.

(i) Prove that $\mathcal{B}$ is a synthetic basis. [5]

(ii) Prove that the set $\mathbf{E}$ of even positive integers is closed in $\{\mathbf{N}, \mathcal{T}\}$, where $\mathcal{T}$ is the topology on $\mathbf{N}$ arising from $\mathcal{B}$. [2]

(iii) (a) Show that 2 is a limit point of $\{1, 3, 5\}$ in $\{\mathbf{N}, \mathcal{T}\}$. [2]

(b) Write down the closure of $\{1, 3, 5\}$ in $\{\mathbf{N}, \mathcal{T}\}$. [No justification is needed.] [2]

Question 22

(i) For each positive integer n, let

$$A_n = \left\{(x_1, x_2) \in \mathbf{R}^2 : |x_1| + |x_2| = \frac{1}{n}\right\}$$

and let

$$A = \bigcup_{n \geq 1}^{\infty} A_n, \quad B = A \cup \{(0, 0)\}.$$

(a) Sketch the sets A_1, A_2 and A_3. [1]

(b) For each of the subsets A and B of $\mathbf{R}^2$ with the usual topology, state whether or not it is compact, briefly justifying your answer. [3]

(ii) Let $\mathcal{T}_1$ be the topology on the set $\mathbf{N}$ of positive integers consisting of $\emptyset$ and the sets $\{n \in \mathbf{N} : n \geq k\}$ for each positive integer k, and let $T_1 = \{\mathbf{N}, \mathcal{T}_1\}$. [You need not verify that $\mathcal{T}_1$ is a topology.] Prove that T_1 is a compact space. [3]

(iii) Let $\mathcal{T}_2$ be the topology on the set $\mathbf{N}$ of positive integers with the sets $\{1, 2, \ldots, k\}$, for each positive integer k, as a basis, and let $T_2 = \{\mathbf{N}, \mathcal{T}_2\}$. [You need not verify that these sets do form a basis.] State whether or not T_2 is a compact space and justify your answer. [4]