M824 Solutions to the Specimen Examination Paper

Question 1

(i) To show that ϕ is a vector space homomorphism, we must show that

$$\phi(\lambda_1 q_1 + \lambda_2 q_2) = \lambda_1\phi(q_1) + \lambda_2\phi(q_2) \text{ for } \lambda_1, \lambda_1 \in \mathbb{C}, q_1, q_2 \in \mathbb{H}.$$

Writing $q_a = x_a + iy_a + jz_a + kw_a$ $\quad (a = 1, 2)$
we have $q_a = (x_a + iy_a) + (z_a + iw_a)j$
so that $u_a = x_a + iy_a \; v_a = z_a + iw_a$ $\quad (a = 1, 2)$.

Thus $\lambda_1\phi(q_1) + \lambda_2\phi(q_2) = \lambda_1(u_1, v_1) + \lambda_2(u_2, v_2)$
$= (\lambda_1 u_1 + \lambda_2 u_2, \lambda_1 v_1 + \lambda_2 v_2)$;
and $\phi(\lambda_1 q_1 + \lambda_2 q_2) = \phi(\lambda_1 x_1 + \ldots + \lambda_2 x_2 + .. + \lambda_2 kw_2)$
$= \phi(\lambda_1 x_1 + \lambda_2 x_2 + i\lambda_1 y_1 + i\lambda_2 y_2 + \lambda_1 jz_1 + \lambda_2 jz_2 + \lambda_1 kw_1 + \lambda_2 kw_2)$
$= \phi((\lambda_1 x_1 + \lambda_2 x_2 + i\lambda_1 y_1 + i\lambda_2 y_2) + (\lambda_1 z_1 + \lambda_2 z_2 + \lambda_1 w_1 i + \lambda_2 w_2 i)j)$
$= (\lambda_1 x_1 + \lambda_2 x_2 + i\lambda_1 y_1 + i\lambda_2 y_2, \lambda_1 z_1 + \lambda_2 z_2 + i\lambda_1 w_1 + i\lambda_2 w_2)$
$= (\lambda_1 u_1 + \lambda_2 u_2, \lambda_1 v_1 + \lambda_2 v_2)$.

So $\phi(\lambda_1 q_1 + \lambda_2 q_2) = \lambda_1\phi(q_1) + \lambda_2\phi(q_2)$, as required.

$\mathbb{H}$ is two-dimensional as a vector space over $\mathbb{C}$ (basis (1,0) and (0,1)).

(ii) Let $q = u + vj$, $q_1 = u_1 + v_1 j$;
then $qq_1 = (u + vj)(u_1 + v_1 j)$
$= uu_1 + uv_1 j + v\overline{u}_1 j + v\overline{v}_1(jj)$
(using $vj = j\overline{v}$ for any $v \in \mathbb{C}$).
$= (uu_1 - v\overline{v}_1) + (uv_1 + v\overline{u}_1)j$.
Thus $\phi(qq_1) = (uu_1 - v\overline{v}_1, uv_1 + v\overline{u}_1)$.
so $\phi(q)\psi(q_1) = (u, v)\begin{bmatrix} u_1 & v_1 \\ -\overline{v}_1 & \overline{u}_1 \end{bmatrix}$ by inspection.

Thus $\psi(q_1) \equiv \begin{bmatrix} u_1 & v_1 \\ -\overline{v}_1 & \overline{u}_1 \end{bmatrix}$.

 SUP 16520 3
2.1