Question 3

(i) If $t \to e^{tA}$ is a one parameter subgroup of $SL(n, \mathbb{R})$, then $e^{tA} \in SL(n, \mathbb{R})$; so $\det e^{tA} = 1$. But $\det e^{tA} = e^{tr\ tA}$

So $e^{tr\ tA} = 1 \Rightarrow tr\ tA = 0 \Rightarrow tr\ A = 0$.

(ii) Let A, $B \in sl(n, \mathbb{R})$. We show that

$$\lambda A + \mu B \in sl(n, \mathbb{R}) \qquad \text{for } \lambda, \mu \in \mathbb{R}$$

in order to see that $sl(n, \mathbb{R})$ is a vector space. The $n \times n$ real matrix $\lambda A + \mu B$ has zero trace, for $tr(\lambda A + \mu B) = \lambda\ tr A + \mu\ tr\ B = 0$ from the linearity properties of trace (Curtis 54) which is all we need to prove.

To find the dimension of $sl(n, \mathbb{R})$ we exhibit a basis;

$$e_{ij}(i, j = 1, \ldots, n)(i \neq j)$$

where each matrix has a "one" in the (i, j) position and zero elsewhere;

$$\lambda_i = \text{diag}(0, 0, \ldots 1, -1, 0, \ldots, 0) \quad (i = 1, \ldots, n-1)$$

where each matrix λ_i is diagonal, with 1 in the (i, i) position, -1 in the $(i+1, i+1)$ position, and zero elsewhere.

Then any matrix $A \equiv (a_{ij} \in sl(n, \mathbb{R})$ may be written

$$A = \sum_{i,j=1}^{n} a_{ij} e_{ij} + \sum b_j \lambda_j;$$

where $b_j = \sum_{i=1}^{j} a_{ii}$; so $\{e_{ij}, \lambda_j\}$ spans $sl(n, \mathbb{R})$.

Further, the set $\{e_{ij}, \lambda_j\}$ is linearly independent; for if

$$\sum a_{ij} e_{ij} + \sum b_j \lambda_j = 0$$

that is, $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & \cdots \\ & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} = 0$ then each element $a_{ij} = 0$.

Thus $\{e_{ij}, \lambda_j\}$ is a basis; and as there are $(n^2 - n) e_{ij}$'s and $(n-1)\lambda_j$'s, there are $(n^2 - 1)$ elements in the basis. So the dimension of $sl(n, \mathbb{R})$ is $n^2 - 1$.