Question 5

(i) Let $\gamma(t)$ be a smooth curve in G, such that $\gamma(0) = I$, then $\gamma'(0)$ is tangent vector to G at the identity I, and we define the map $d\phi : T_eG \to T_eH$ by $\quad d\phi(\gamma'(0)) = (\phi \circ \gamma)'(0)$.

(ii) Choose as basis for $T_eGL(2,\mathbb{R})$

$$\{e_1, e_2, e_3, e_4\} \equiv \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Let $\gamma(t) = \begin{bmatrix} a_1(t) & a_2(t) \\ a_3(t) & a_4(t) \end{bmatrix}$ be a smooth curve in $GL(2,\mathbb{R})$

such that $\gamma(0) = \begin{bmatrix} a_1(0) & a_2(0) \\ a_3(0) & a_4(0) \end{bmatrix} = I$.

Consider $\phi : G \to \mathbb{R} \qquad (\phi \equiv \det)$.

Then $d\phi(\gamma'(0)) = (\phi \circ \gamma)'(0)$; by definition.

That is, $d\phi(\sum a_i'(0)e_i) = (a_1(t)a_4(t) - a_2(t)a_3(t))'(0)$

$$= a_1'(0)a_4(0) + a_1(0)a_4'(0) - a_2'(0)a_3(0) - a_2(0)a_3'(0)$$
$$= a_1'(0) + a_4'(0) \quad (\text{since } a_2(0) = a_3(0) = 0).$$

In other words $[a_1'(0), a_2'(0), a_3'(0), a_4'(0)] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = a_1'(0) + a_4'(0)$

so the matrix for $d\phi = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(Alternatively; we could consider $d\phi$ as the trace of $\gamma'(0)$).

Question 6

(i) Consider $q \in \mathbb{H}$; $q = x + yj + zj + wk$.

Then $\quad \psi(q) = \psi(x + yi + zj + wk)$

$= x + y\psi(i) + z\psi(j) + w\psi(ij)$

since ψ is a (linear) algebra homomorphism.

$= x + y\psi(i) + z\psi(j) + w\psi(i)\psi(j)$

since ψ is a (multiplicative) algebra homomorphism.

$= x + ye_1 + ze_2 + we_1e_2$ by definition of ψ

which belongs to C_2.

Now ψ *is* a map *onto* C_2; since for x, y, z, $w \in \mathbb{R}$ all elements of C_2 are obtained. Further, ψ is injective - for given $0 \in C_2$, $\psi^{-1}(0) = 0$ [only $0 \in \mathbb{H}$ maps to $0 \in C_2$].

Therefore, since ψ is given as an algebra homorphism, it is an isomorphism.

(ii) We have that $\cos\theta + \sin\theta k \xrightarrow{\psi} \cos\theta + \sin\theta e_1e_2 \xrightarrow{\phi} \cos\theta + \sin\theta e_1e_2$.

Writing $(\cos\theta + \sin\theta e_1e_2) = (\cos\frac{\theta}{2}e_1 + \sin\frac{\theta}{2}e_2)(-\cos\frac{\theta}{2}e_1 + \sin\frac{\theta}{2}e_2)$, we see that the image $\cos\theta + \sin\theta e_1e_2 \in C_3$ is an even product of elements in Pin(3), and therefore belongs to Spin(3).