Question 2

(i) Firstly, the map is from $U(2)$ to $U(4)$:

because $\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}^{+} = \begin{bmatrix} uu^{\dagger} & 0 \\ 0 & uu^{\dagger} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$,
so that $\phi(u)$ is unitrary.

Further, ϕ is a homomorphism; for

$$\phi(u_1)\phi(u_2) = \begin{bmatrix} u_1 & 0 \\ 0 & u_1 \end{bmatrix} \begin{bmatrix} u_2 & 0 \\ 0 & u_2 \end{bmatrix} = \begin{bmatrix} u_1u_2 & 0 \\ 0 & u_1u_2 \end{bmatrix} = \phi(u_1u_2)$$

so that $\phi(u_1)\phi(u_2) = \phi(u_1u_2)$.

Finally, ϕ is clearly injective; if $\phi(u) = I$, then $u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(ii) We note that the given matrix is of the form $\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$,
with $u = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$
(on diagonalizing).

That is, $u = v\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} v^{\dagger}$ with $v = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \in U(2)$.

Therefore it suffices to find a path γ in $U(2)$ from $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ to $u^{\dagger}$; one such is
given by $\gamma(t) = v\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2t} \end{bmatrix} v^{\dagger}$; clearly γ is smooth (differentiable); and
$\gamma(t) \in U(2)$.

Then $\phi \circ \gamma : \mathbb{R} \to U(4)$ is the required path in $U(4)$.

(Explicitly $\phi \circ \gamma(t) = \frac{1}{2}\left[\begin{array}{cc|cc} 1+e^{i\pi/2t} & 1-e^{i\pi/2t} & 0 & 0 \\ 1-e^{i\pi/2t} & 1+e^{i\pi/2t} & 0 & 0 \\ \hline 0 & 0 & 1+e^{i\pi/2t} & 1-e^{i\pi/2t} \\ 0 & 0 & 1-e^{i\pi/2t} & 1+e^{i\pi/2t} \end{array}\right]$