Question 1

Write the quaternion $q \in \mathbb{H}$ as

$$q = x + iy + jz + kw = u + vj \qquad (x, y, z, w \in \mathbb{R}) \quad (u, v \in \mathbb{C})$$

(i) Show that the map $\phi\colon \mathbb{H} \longrightarrow \mathbb{C}^2$
$q \longrightarrow (u, v)$
is a linear isomorphism of vector spaces (over $\mathbb{C}$).
What is the dimension of $\mathbb{H}$ as a vector space over $\mathbb{C}$?

(ii) Define a map $\Psi\colon \mathbb{H} \longrightarrow M_2(\mathbb{C})$
by $\quad \phi(qq_1) = \phi(q)\ \Psi(q_1) \quad (q, q_1 \in \mathbb{H})$.
When $\quad q_1 = x_1 + iy_1 + jz_1 + kw_1$,
write $\Psi(q_1)$ explicitly as a 2×2 matrix in the components of q_1.

Question 2

(i) Show that the map $\phi\colon u \longmapsto \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ defines an injective homomorphism from $U(2)$ to $U(4)$. [u, 0 are 2×2 matrices.]

(ii) Find a path in $U(4)$ from

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{to} \quad \frac{1}{2}\begin{bmatrix} 1+i & 1-i & 0 & 0 \\ 1-i & 1+i & 0 & 0 \\ 0 & 0 & 1+i & 1-i \\ 0 & 0 & 1-i & 1+i \end{bmatrix}.$$

Question 3

The set $SL(n, \mathbb{R})$, is defined by $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det A = 1\}$.

(i) Show that if $t \longmapsto e^{tA}$ is a one-parameter subgroup of $SL(n, \mathbb{R})$, then trace $A = 0$.

(ii) The set $sl(n, \mathbb{R})$ is defined by

$$sl(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \text{trace } A = 0\}.$$

Show that $sl(n, \mathbb{R})$ is a vector space over $\mathbb{R}$, and find its dimension.

Question 4

(i) A subset of $\mathbb{R}^N$ is said to be *compact* if it is closed and bounded. Considering $GL(n, \mathbb{R})$ as a subset of the metric space $\mathbb{R}^{n^2}$ in the usual way, show that $GL(n, \mathbb{R})$ is *not* compact.

(ii) Show that the set $G = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$ is a connected subgroup of $GL(2, \mathbb{R})$.

Question 5

(i) Let $\phi\colon G \longrightarrow H$ be a smooth homomorphism of matrix groups. Define the differential map $d\phi\colon T_eG \longrightarrow T_eH$.

(ii) Consider the smooth homomorphism $\det\colon GL(2, \mathbb{R}) \longrightarrow \mathbb{R}$
$A \longmapsto \det A$.
Choosing a basis for $T_eGL(2, \mathbb{R})$, express $d \det$ as a matrix.