Question 4

(i) We shall show that $GL(n,\mathbb{R})$ is not bounded, and therefore not compact.

Consider the sequence $\{g_r\}$ in $GL(n,\mathbb{R})$, where

$$g_r = \begin{bmatrix} 1 & 0 & .. & r \\ & 1 & 0 & .. \\ 0 & & & 1 \end{bmatrix}$$

where, apart from 1's on the diagonal, the only non-zero element is the integer r in the $(1,n)$ position.

Then since $\det g_r \neq 0$ (in fact $\det g_r = 1$), $g_r \in GL(n,\mathbb{R})$, and so $\{g_r\}$ is a sequence in $GL(n,\mathbb{R})$.

But $\{g_r\}$ is not bounded, for the distance, of g_r from the unit matrix I, say, is given by (using the usual metric in $\mathbb{R}^{n^2}$)

$$d(g_r, I) = r$$

which increases without bound.

(ii) Since the element $g_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ belongs to $GL(2,\mathbb{R})$ ($\det g_a = 1$), $G \subset GL(2,\mathbb{R})$.

Also G is a *subgroup* of $GL(2,\mathbb{R})$; for

$$g_a g_b = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = g_{a+b} \in G.$$

and $I (\equiv g_o) \in G$; $(g_a)^{-1} = g_{-a}$ also belongs to G.

We now exhibit a path γ in G from I to any element g_a.

For example $\gamma(t) = g_{at}$ will do;

$\gamma(t)$ is smooth (all elements of g_{at} are differentiable),

$\gamma(0) = I$ and $\gamma(1) = g_a$.

Therefore G is connected.