Question 8

If T is a standard maximal torus in G, the roots of G are computed by considering the adjoint action,

$$Ad(t)e_i \qquad (t \in T)$$

where $\{e_i\}$ is a basis for $L(G)$.

Given $\gamma_i(u)$ a one-parameter subgroup in G such that $\gamma_i'(0) = e_i$, $Ad(t)$ is the differential of the map

$$A(t) : g \to tgt^{-1},$$

so $$Ad(t)(\gamma_i'(0)) = (t\gamma_i(u)t^{-1})' = te_it^{-1}.$$

We may thus directly evaluate the $Ad(t)(e_i)$ on the given basis $\{e_i\}$. Writing an element $t \in T$ as

$$t = \begin{bmatrix} e^{i\theta_1} & & \\ & e^{i\theta_2} & \\ & & e^{i\theta_3} \end{bmatrix}, \text{ where } \theta_3 = -(\theta_1 + \theta_2)$$

we compute $te_1t^{-1} = e_1$, $te_2t^{-1} = e_2$.

$$\text{Now } te_3t^{-1} = \begin{bmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & e^{i(\theta_1-\theta_2)} & 0 \\ -e^{-i(\theta_1-\theta_2)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \cos(\theta_1 - \theta_2)e_3 + \sin(\theta_1 - \theta_2)e_6$$

$$\text{Similarly } te_4t^{-1} = \begin{bmatrix} 0 & 0 & -e^{-i(\theta_3-\theta_1)} \\ 0 & 0 & 0 \\ e^{i(\theta_3-\theta_1)} & 0 & 0 \end{bmatrix}$$ (by inspection-analogy with first calculation!)

$$= \cos(\theta_3 - \theta_1)e_4 + \sin(\theta_3 - \theta_1)e_7.$$

Similarly $te_5t^{-1} = \cos(\theta_2 - \theta_3)e_5 + \sin(\theta_2 - \theta_3)e_8$.

$$\text{Also} \quad te_6t^{-1} = \begin{bmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{bmatrix} \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{bmatrix}$$

$$= \cos(\theta_1 - \theta_2)e_6 - \sin(\theta_1 - \theta_2)e_3$$

$$te_7t^{-1} = \cos(\theta_3 - \theta_1)e_7 - \sin(\theta_3 - \theta_1)e_4$$

$$te_8t^{-1} = \cos(\theta_2 - \theta_3)e_8 - \sin(\theta_2 - \theta_3)e_5.$$

Thus the roots are $\theta_1 - \theta_2$, $\theta_3 - \theta_1 \equiv -2\theta_1 - \theta_2$, $\theta_2 - \theta_3 \equiv 2\theta_2 + \theta_1$.

There are only two independent roots; these may be taken to be $\phi_1 \equiv \theta_1 - \theta_2$ and $\phi_2 = \theta_1 + 2\theta_2$ (so that $(2\theta_1 + \theta_2) = \phi_1 + \phi_2$).