Question 7

(i) The normalizer N of T in $U(2)$ is defined by

$$N(T) = \{x \in U(2) | xTx^{-1} = T\}$$

Given $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U(2)$, we require that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{bmatrix} \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} = \begin{bmatrix} e^{i\theta_1'} & 0 \\ 0 & e^{i\theta_2'} \end{bmatrix}$$

i.e. $$\begin{bmatrix} e^{i\theta_1}|a|^2 + e^{i\theta_2}|b|^2 & e^{i\theta_1} a\bar{c} + e^{i\theta_2} b\bar{d} \\ e^{i\theta_1}\bar{a}c + e^{i\theta_2}\bar{b}d & e^{i\theta_1}|c|^2 + e^{i\theta 2}|d|^2 \end{bmatrix} = \begin{bmatrix} e^{i\theta_1'} & 0 \\ 0 & e^{i\theta_2'} \end{bmatrix}$$

for all θ_1, θ_2, some θ_1', θ_2'.

Thus, $a\bar{c} = 0$, $b\bar{d} = 0$.

Because $x \in U(2)$ we cannot have $a = 0$, $b = 0$ together or $c = 0$, $d = 0$ together; so the only possibilities are $a = 0$, $d = 0$ or $b = 0$, $c = 0$.

Thus $x = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ or $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$.

$|a| = |d| = 1 \qquad |b| = |c| = 1$

Thus $N = \{\begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{bmatrix}, \begin{bmatrix} 0 & e^{i\theta_1} \\ e^{i\theta_2} & 0 \end{bmatrix} | \theta_1, \theta_2 \in \mathbb{R}\}$

$= T \cup \{\begin{bmatrix} 0 & e^{i\theta_1} \\ e^{i\theta_2} & 0 \end{bmatrix} | \theta_1, \theta_2 \in \mathbb{R}\}$.

(ii) The Weyl group W is defined by $W = N/T$.

Writing $N/T = \{T, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} T\}$

(since any $\begin{bmatrix} 0 & e^{i\theta_1} \\ e^{i\theta_2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{i\theta_2} & 0 \\ 0 & e^{i\theta_1} \end{bmatrix}$)

we see that $N/T \simeq \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\} \simeq Z_2$.

(ii) To show that $U(2)$ splits, we must exhibit a homomorphism $\gamma : W \to N$ so that $\beta \circ \gamma$ is the identity on W, in the sequence $T \xrightarrow{\alpha} N \underset{\overleftarrow{\gamma}}{\xrightarrow{\beta}} W$

Define $\gamma : W \to N$ by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in N$,

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in N$.

It is clear that γ is a homomorphism, and that

$$\beta \circ \gamma : W \to W$$

is the identity on W.