Question 6

(i) Show that the algebra homomorphism

$$\Psi : \mathbb{H} \longrightarrow C_2$$

defined by $\Psi(i) = e_1$, $\Psi(j) = e_2$ gives an isomorphism of the real quaternion algebra $\mathbb{H}$ (generated by i, j, k) and the real Clifford algebra C_2 (generated by e_1, e_2).

(ii) Define the algebra homomorphism $\phi : C_2 \longrightarrow C_3$ by

$$\phi(e_i) = e_i e_3 \qquad (i = 1, 2).$$

Consider the sequence of maps

$$Sp(1) \longrightarrow \mathbb{H} \xrightarrow{\Psi} C_2 \xrightarrow{\phi} C_3.$$

Find the image of $\cos\theta + \sin\theta\, k \in Sp(1)$ under this sequence, and show that it belongs to Spin (3).

Question 7

The group $U(2)$ is the compact connected Lie group of 2×2 complex matrices satisfying $UU^\dagger = I$, (where I is the unit matrix, and $U^\dagger \equiv {}^t\overline{U}$). A standard maximal torus T in $U(2)$ is given by

$$T = \left\{ \begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{bmatrix} \quad \Big| \ \theta_1, \theta_2 \in \mathbb{R} \right\}.$$

(i) Find the Normalizer N of T in $U(2)$.

(ii) Define the Weyl group W of $U(2)$, and show that W is isomorphic to Z_2 (the two-element group).

(iii) Show that $U(2)$ splits (i.e. that the sequence $T \xrightarrow{\alpha} N \xrightarrow{\beta} W$ splits).

Question 8

Consider the group $SU(3)$, with standard maximal torus

$$T = \left\{ \begin{bmatrix} e^{i\theta_1} & & \\ & e^{i\theta_2} & \\ & & e^{-i(\theta_1+\theta_2)} \end{bmatrix} \right\}.$$

Writing a basis for the Lie Algebra $su(3)$ of $SU(3)$ as

$$e_1 = \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad e_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad e_6 = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e_7 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad e_8 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

compute the roots of $SU(3)$.
How many independent roots are there?

[END OF QUESTION PAPER]