Question 5

(i) Using classification procedures from page 25 of the Handbook:

$$B^2 - 4AC = 0^2 - 4 \times 1 \times x = -4x.$$

The equation is hyperbolic when $x < 0$, so $R = \{(x, y) : x < 0, -\infty < y < \infty\}$, or is the left-hand half-plane, excluding the y-axis. $1\frac{1}{2}$

(ii) Two. $\frac{1}{2}$

(iii) Slopes, s, of characteristic curves are given by (see page 25 of the Handbook)

$$s^2 + x = 0,$$

so $\quad s = +\sqrt{-x}$ or $-\sqrt{-x}$.

Putting $s = \dfrac{dy}{dx}$, we have

$$\frac{dy}{dx} = \sqrt{-x} \quad \text{or} \quad \frac{dy}{dx} = -\sqrt{-x}.$$

So $\quad y = -\frac{2}{3}(-x)^{3/2} + C \quad$ or $\quad y = \frac{2}{3}(-x)^{3/2} + C$.

Thus we may choose $\quad \zeta = 3y + 2(-x)^{3/2} \quad$ and $\quad \phi = 3y - 2(-x)^{3/2}$. 4

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Question 6

(i) Substitute $\phi = f(z)\cos(kx - \omega t)$ in

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

Then $(f''(z) - k^2 f(z))\cos(kx - \omega t) = 0$, where the dash represents differentiation with respect to z.

Solution of $f'' - k^2 f = 0$ is of the form $f(z) = Ae^{-kz} + Be^{kz}$ (A, B constants), and so $\phi(x, z, t) = (Ae^{-kz} + Be^{kz})\cos(kx - \omega t)$. $2\frac{1}{2}$

Since ϕ remains finite as $z \to -\infty$,

$$A = 0.$$

From $\dfrac{\partial^2 \phi}{\partial t^2} + g\dfrac{\partial \phi}{\partial z} = 0$, we have

$$Be^{kz}\cos(kx - \omega t)(-\omega^2 + gk) = 0.$$

Thus $\omega^2 = gk$, for non-trivial solutions. 2

(ii) The wave speed c is $\dfrac{\omega}{k}$, so

$$c^2 = \frac{\omega^2}{k^2} = \frac{gk}{k^2} = \frac{g}{k}.$$

Wavelength λ is given by $k\lambda = 2\pi$, therefore

$$k = \frac{2\pi}{\lambda} \quad \text{and} \quad c^2 = \frac{g\lambda}{2\pi}.$$ 2

(iii) From page 33 of the Handbook, the group speed is

$$\begin{aligned} c_g &= c + k\frac{dc}{dk} \\ &= c + k\frac{d}{dk}\left(\sqrt{\frac{g}{k}}\right) \\ &= c - \tfrac{1}{2}k\frac{\sqrt{g}}{k^{3/2}} \\ &= c - \tfrac{1}{2}\sqrt{\frac{g}{k}} = \tfrac{1}{2}c. \end{aligned}$$ $1\frac{1}{2}$

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