Question 11

(i) The one-dimensional wave equation (Handbook, page 31) is $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$.

Setting $u(x,t) = X(x)T(t)$, it becomes 1

$$X''(x)T(t) = \frac{1}{c^2}X(x)T''(t).$$

Separating variables and applying the usual argument we have

$$X'' + \lambda X = 0$$
$$T'' + \lambda c^2 T = 0. \qquad 2$$

The boundary conditions are

$$u(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(l,t) = 0, \quad t \geq 0, \qquad 2$$

i.e. $X(0) = X'(l) = 0$. 1

By Theorem 4 of *Unit 11* (Handbook, page 29) eigenvalues of problem for X are only positive ones. When $\lambda > 0$,

$$X(x) = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$$

and $X(0) = 0 \Longrightarrow B = 0$.

$$X'(x) = \sqrt{\lambda}A\cos\sqrt{\lambda}x$$

and $X'(l) = 0 \Longrightarrow \sqrt{\lambda}A\cos\sqrt{\lambda}l = 0$.

If $A \neq 0$, $\sqrt{\lambda}l = \pm(n+\frac{1}{2})\pi \quad n = 0,1,2,\ldots$

Eigenvalues are $\lambda_n = \dfrac{(n+\frac{1}{2})^2\pi^2}{l^2} \quad n = 0,1,2,\ldots$

and corresponding eigenfunctions are $\sin\dfrac{(n+\frac{1}{2})\pi x}{l}, \quad n = 0,1,2,\ldots$ 5

General solution for T for $\lambda > 0$ is

$$T(t) = C\sin\sqrt{\lambda}ct + D\cos\sqrt{\lambda}ct.$$

Combining solutions for X and T, we have

$$u_n(x,t) = \left(c_n\sin\frac{(n+\frac{1}{2})\pi ct}{l} + d_n\cos\frac{(n+\frac{1}{2})\pi ct}{l}\right)\sin\frac{(n+\frac{1}{2})\pi x}{l},$$
$$n = 0,1,2,3,\ldots$$

Taking a linear combination of these:

$$u(x,t) = \sum_{n=0}^{\infty}\left(c_n\sin\frac{(n+\frac{1}{2})\pi ct}{l} + d_n\cos\frac{(n+\frac{1}{2})\pi ct}{l}\right)\sin\frac{(n+\frac{1}{2})\pi x}{l}. \qquad 3$$

(ii) Since $u(x,0) = 0$ we have

$$0 = \sum_{n=0}^{\infty} d_n\sin\frac{(n+\frac{1}{2})\pi x}{l} \Longrightarrow d_n = 0 \quad n = 0,1,2,\ldots \qquad 1$$

(Fourier Series of zero function)

So $u(x,t) = \displaystyle\sum_{n=0}^{\infty} c_n\sin\frac{(n+\frac{1}{2})\pi ct}{l}\sin\frac{(n+\frac{1}{2})\pi x}{l}$.

Assuming term by term differentiation, we have

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=0}^{\infty}\frac{(n+\frac{1}{2})\pi c}{l}c_n\cos\frac{(n+\frac{1}{2})\pi ct}{l}\sin\frac{(n+\frac{1}{2})\pi x}{l}. \qquad 1$$

Applying the initial condition,

$$v\sin\left(\frac{\pi x}{2l}\right) = \sum_{n=0}^{\infty}\left(\frac{n+\frac{1}{2}}{l}\right)c\pi\, c_n\sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l}$$

By inspection,

$$c_0\left(\frac{\pi c}{2l}\right) = v, \quad \text{so that } c_0 = \frac{2lv}{\pi c} \qquad 2$$

and $\quad c_n = 0, \quad n = 1,2,3,\ldots$

(Alternatively use Fourier Series but this is long.)