Question 9

(i) The velocity vanishes when $r = a$, $r = 2a$, i.e. 1

$$\mathbf{u}(a, \theta, z) = \mathbf{u}(2a, \theta, z) = \mathbf{0},$$

referred to cylindrical polar coordinates r, θ, z, where the z-axis is along the common axis.

(ii) The fluid is Newtonian and incompressible and of constant viscosity so that the incompressible Navier–Stokes equations hold.

The flow is steady, so '$\dfrac{\partial}{\partial t} = 0$' and, by symmetry, all quantities are independent of θ, so '$\dfrac{\partial}{\partial \theta} = 0$'.

Assume that $u_r = u_\theta = 0$, since they vanish on the cylinder surface and the primary flow is along the z-axis.

Assume that body forces are zero, i.e.

$$F_r = F_\theta = F_z = 0.$$ 5

(iii) The equation of continuity (see Section 3, page 18 of the Handbook) is

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0,$$

and this reduces to

$$\frac{\partial u_z}{\partial z} = 0$$

using the above assumptions.

Thus u_z is independent of z (and θ, t), and therefore

$$\mathbf{u} = u_z(r)\,\mathbf{e}_z.$$ 2

(iv) The Navier–Stokes equations reduce to

$$0 = -\frac{\partial p}{\partial r}, \qquad (1)$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta}, \qquad (2)$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{d^2 u_z}{dr^2} + \frac{1}{r}\frac{du_z}{dr}\right), \qquad (3)$$

where μ is the coefficient of viscosity.

Since p does not depend upon r, θ (from (1) and (2)) or t (from the fact that $\dfrac{\partial}{\partial t} = 0$),

$$p = f(z).$$

Substituting this in (3),

$$0 = \underbrace{-f'(z)}_{\substack{\text{depends}\\ \text{on } z}} + \underbrace{\mu\left(\frac{d^2 u_z}{dr^2} + \frac{1}{r}\frac{du_z}{dr}\right)}_{\text{depends on } r}.$$

Both quantities must be constant, C.

$\dfrac{dp}{dz}$ must be negative, since p decreases in the direction of increasing z.

Therefore $-\dfrac{dp}{dz} = C = \dfrac{p_0 - p_1}{l}$, where $C > 0$, and (after a little rearranging) (3) becomes

$$\frac{d^2 u_z}{dr^2} + \frac{1}{r}\frac{du_z}{dr} = \frac{p_1 - p_0}{l\mu}.$$ 7