For the second integral,

$$x = 1,\ y = z = \beta, \qquad \mathbf{r} = \mathbf{i} + \beta\mathbf{j} + \beta\mathbf{k}, \quad 0 \le \beta \le 1,$$

and

$$\begin{aligned}\int_{C_2} \mathbf{v} \cdot d\mathbf{r} &= \int_0^1 \mathbf{v} \cdot \frac{d\mathbf{r}}{d\beta}\, d\beta \\ &= \int_0^1 [(2\beta + \beta + 2)\mathbf{i} + (1 + 2\beta)\mathbf{j} + \mathbf{k}] \cdot (\mathbf{j} + \mathbf{k})\, d\beta \\ &= \int_0^1 (1 + 2\beta + 1)\, d\beta = \left[\beta^2 + 2\beta\right]_0^1 = 3.\end{aligned}$$

For the third integral,

$$x = y = z = \gamma, \qquad \mathbf{r} = \gamma\mathbf{i} + \gamma\mathbf{j} + \gamma\mathbf{k}, \quad 0 \le \gamma \le 1,$$

and

$$\begin{aligned}\int_{C_3} \mathbf{v} \cdot d\mathbf{r} &= \int_1^0 \mathbf{v} \cdot \frac{d\mathbf{r}}{d\gamma}\, d\gamma \\ &= \int_1^0 [(2\gamma^2 + \gamma + 2\gamma)\mathbf{i} + (\gamma + 2\gamma)\mathbf{j} + \gamma^2\mathbf{k}] \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\, d\gamma \\ &= \int_1^0 (2\gamma^2 + 3\gamma + 3\gamma + \gamma^2)\, d\gamma = \left[\gamma^3 + 3\gamma^2\right]_1^0 = -4.\end{aligned}$$

Thus $\oint_C \mathbf{v} \cdot d\mathbf{r} = 1 + 3 - 4 = 0.$ 7

Now $$\operatorname{curl} \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz + y + 2x & x + 2y & x^2 \end{vmatrix} = \mathbf{i}(0 - 0) - \mathbf{j}(2x - 2x) + \mathbf{k}(1 - 1) = \mathbf{0}.$$

From Stokes' Theorem, circulation around all closed curves is zero at time $t = 0$. 3

To deduce that the circulation is zero around all curves, we should need to know that the fluid is incompressible and inviscid and is moving under the action of conservative forces (Kelvin's Theorem; see page 22 of the Handbook). 2

(ii) $\boldsymbol{\nabla}\phi = (2xz + y + 2x)\mathbf{i} + (x + 2y)\mathbf{j} + x^2\mathbf{k}$ gives

$$\frac{\partial \phi}{\partial x} = 2xz + y + 2x, \tag{a}$$

$$\frac{\partial \phi}{\partial y} = x + 2y, \tag{b}$$

$$\frac{\partial \phi}{\partial z} = x^2. \tag{c}$$

From (a), $\phi(x, y, z) = x^2 z + xy + x^2 + f(y, z)$.

Substituting for ϕ in (b),

$$\frac{\partial \phi}{\partial y} = x + \frac{\partial f}{\partial y} = x + 2y,$$

so that $f = y^2 + g(z)$.

Therefore $\phi(x, y, z) = x^2 z + xy + x^2 + y^2 + g(z)$.

Substituting for ϕ in (c),

$$x^2 + g'(z) = x^2,$$

so that $g'(z) = 0$ and $g(z) = A$, a constant.

Therefore $\phi(x, y, z) = x^2 z + xy + x^2 + y^2 + A$, and

$$\begin{aligned}\int_D^E \mathbf{v} \cdot d\mathbf{r} &= \phi(E) - \phi(D) = \phi(1, 1, 0) - \phi(1, 0, 1) \\ &= 3 + A - 2 - A = 1.\end{aligned}$$ 7

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