(ii) In the notation of Theorem 4 (page 29 of the Handbook), we have

$$q(x) \geq 0, \quad \alpha_1, \alpha_2 \geq 0 \quad \text{and} \quad \beta_1, \beta_2 \geq 0.$$

In fact, $q(x) = 0$, $\alpha_1 = \beta_1 = 0$.

Then by Theorem 4, all eigenvalues are non-negative, and zero is an eigenvalue with eigenfunction 1.

Consider $X''(x) + \lambda X(x) = 0 \quad (\lambda > 0)$.

Then $X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$.

$X'(0) = 0$ and $X'(\pi) = 0$ give

$$B = 0,$$
$$A\sin\sqrt{\lambda}\pi = 0.$$

Then $\sqrt{\lambda} = n$ $(n \geq 1)$, or $\lambda = n^2$ $(n \geq 1)$.

Corresponding eigenfunctions are $\cos nx$ $(n \geq 1)$.

Eigenvalues are $\lambda_n = n^2$ $(n = 0, 1, 2, \ldots)$
with eigenfunctions $\cos nx$ $(n = 0, 1, 2, \ldots)$. 5

(iii) From page 31 of the Handbook,

$$r^2\nabla^2 u = r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Put $u(r,\theta) = R(r)X(\theta)$ to obtain

$$r^2R''X + rR'X + RX'' = 0.$$

Then $\dfrac{r^2R'' + rR'}{R} = \dfrac{-X''}{X} = \lambda$, constant.

Therefore $\quad r^2R'' + rR' - \lambda R = 0,$
and $\quad X'' + \lambda X = 0$ with $X'(0) = X'(\pi) = 0$. 2

From part (ii), we note that $\lambda = n^2$ $(n = 0, 1, 2\ldots)$.

We solve the equations for R with $\lambda = n^2$.

Putting $R = r^\alpha$ gives

$$(\alpha(\alpha-1) + \alpha - n^2)r^\alpha = 0$$

or $\quad \alpha^2 - n^2 = 0 \quad$ (since $r \neq 0$).

Then $R_0(r) = A_0 + B_0\log_e r \quad (n = 0)$,
$\quad R_n(r) = A_n r^n + B_n r^{-n} \quad (n > 0)$.

Boundedness condition at $r = 0$ gives $B_n = 0$ $(n = 0, 1, 2, \ldots)$. 4

Hence $u_n(r,\theta) = R_n(r)X_n(\theta) = A_n r^n \cos n\theta$ $(n = 0, 1, 2, \ldots)$.

Then $u(r,\theta) = \displaystyle\sum_{n=0}^{\infty} A_n r^n \cos n\theta$ is a solution. On setting $r = 1$ and using the given condition,

$$\sum_{n=0}^{\infty} A_n \cos n\theta = \sin\theta.$$

$|\sin\theta| = \sin\theta$ because $0 < \theta < \pi$.

From part (i),

$$A_n = 0, \quad \text{if } n \text{ is odd},$$
$$A_n = \frac{4}{\pi(1-n^2)}, \quad \text{if } n \text{ is even}.$$

Thus, $\quad u(r,\theta) = \dfrac{2}{\pi} + \dfrac{4}{\pi}\displaystyle\sum_{m=1}^{\infty}\frac{r^{2m}\cos 2m\theta}{1-4m^2}$. 3

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