Question 9

The function $u(x, y)$ satisfies the partial differential equation

$$y^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} - \frac{y^2}{x}\frac{\partial u}{\partial x} - \frac{x^2}{y}\frac{\partial u}{\partial y} = 0 \quad (x \neq 0, y \neq 0). \qquad (2)$$

(i) Show that this equation is parabolic at every point of the (x, y)-plane where it is defined.

Find the equations of the characteristic curves passing through each of these points. Hence show that the characteristic coordinates may be chosen to be

$$\zeta = x^2 + y^2, \quad \phi = y. \qquad [6]$$

(ii) Use the characteristic coordinates in Part (i), and the chain rule, to transform the partial differential equation (2) to the standard form

$$\frac{\partial^2 u}{\partial \phi^2} - \frac{1}{\phi}\frac{\partial u}{\partial \phi} = 0 \qquad (\phi \neq 0). \qquad [10]$$

(iii) Hence find the general solution of the partial differential equation (2) for $x \neq 0$, $y \neq 0$. [4]

Question 10

(i) Show that all the eigenvalues λ_n of the eigenvalue problem

$$X''(x) + \lambda X(x) = 0 \quad (0 < x < \pi),$$
$$X'(0) = X'(\pi) = 0,$$

are given by $\lambda_n = n^2$ $(n = 0, 1, 2, \ldots)$ and the corresponding eigenfunctions are given by $X_n(x) = \cos nx$. [5]

(ii) A porous rod of length π containing moisture has both ends sealed and loses moisture through its surface to dry air. The concentration of moisture $u(x, t)$ satisfies the problem given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} + \gamma^2 u \quad (0 < x < \pi, t > 0),$$

with

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0 \quad (t \geq 0),$$

and

$$u(x, 0) = \sin^2 x \quad (0 < x < \pi),$$

where x and t represent distance along the rod and time, respectively, and $k(> 0)$ and γ are constants.

(a) Use a trigonometric identity to express the given initial condition in terms of the cosine function.

(b) Use the method of separation of variables and the result of Part (i) to determine the concentration of moisture in the rod at times subsequent to $t = 0$. [15]