

1/1) Let $u = x^\lambda$ so $\lambda(\lambda-1) - \lambda + 1 = 0$ or $(\lambda-1)^2 = 0$

General solution $u = (A + B \log_e x) x$

$u(1) = 0 \Rightarrow A = 0$ $u(e) = e \Rightarrow B = 1$ $u = x \log_e x$

ii) $\frac{\partial u}{\partial x} - \frac{1}{x} u = 0$, IF = $\frac{1}{x}$, $\frac{\partial}{\partial x}(\frac{u}{x}) = 0$, $u = x f(x)$

2/ Assume $F = k d^\alpha n^\beta v^\gamma \rho^\delta \mu^\epsilon$ $[L] = ML^{-1}T^{-1}$
 $[F] = MLT^{-2}$ $[d] = L$ $[n] = T^{-1}$ $[v] = LT^{-1}$ $[\rho] = ML^{-3}$
 $MLT^{-2} = L^\alpha T^{-\beta} L^\gamma T^{-\gamma} M^\delta L^{-3\delta} M^\epsilon L^{-\epsilon} T^{-\epsilon}$

Equating powers $1 = \delta + \epsilon$, $1 = \alpha + \gamma - 3\delta - \epsilon$

$-2 = -\beta - \gamma - \epsilon$ giving $\delta = 1 - \epsilon$, $\gamma = -\beta + 2 - \epsilon$

$\alpha = 2 + \beta - \epsilon$ so $F = k d^{2+\beta-\epsilon} v^{-\beta+2-\epsilon} \rho^{1-\epsilon} \mu^\epsilon$ Hence $\alpha = 2 + \beta - \epsilon$

3/ i) $\nabla \cdot \underline{u} = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) + \frac{1}{r^2} \frac{\partial}{\partial \theta} (-\cos 2\theta) = 0$ \therefore wave eq

ii) $\frac{1}{r} \frac{\partial u}{\partial \theta} = \frac{\sin 2\theta}{r^3} \frac{\partial u}{\partial r} = \frac{\cos 2\theta}{2r^2} \frac{\partial u}{\partial \theta}$
 Int $\textcircled{1}$ $\psi = -\frac{\cos 2\theta}{2r} + f(r)$ Sub in $\textcircled{2} \Rightarrow f'(r) = 0$
 or $f(r) = \text{const}$ $\therefore \psi = -\frac{\cos 2\theta}{2r} + \text{const}$

Streamlines $r = k \cos 2\theta$ where k is a constant

iii) When $r = 1$, $\theta = 0$ so streamline is $r = \cos 2\theta$

Pathlines and streamlines are the same as no time dependence so pathline $w = \cos 2\theta$

4/ i) $\oint_C \underline{u} \cdot d\underline{r} = \int_{OA} \underline{u} \cdot d\underline{r} + \int_{AB} \underline{u} \cdot d\underline{r} + \int_{BO} \underline{u} \cdot d\underline{r}$

$\int_{OA} = \int_0^1 \underline{u} \cdot \frac{d\underline{r}}{dr} dr = \int_0^1 0 dr = 0$

$\int_{AB} = \int_0^1 \underline{u} \cdot \frac{d\underline{r}}{d\beta} d\beta = \int_0^1 (1 + \beta^2) d\beta = [\beta + \frac{\beta^3}{3}]_0^1 = \frac{4}{3}$

$\int_{BO} = \int_1^0 \underline{u} \cdot \frac{d\underline{r}}{d\delta} d\delta = \int_1^0 (2\delta^2 \underline{i} + 2\delta \underline{j}) (\underline{i} + \delta \underline{j}) d\delta = \int_1^0 4\delta^2 d\delta = -\frac{4}{3}$

Hence $\oint_C \underline{u} \cdot d\underline{r} = 0$ so circulation is zero.

ii) $\text{curl } \underline{u} = (0-0) \underline{i} + (2\delta - 2\delta) \underline{j} + (2\alpha - 2\alpha) \underline{k} = 0$

From Stokes' theorem circulation around all closed curves is zero at $t=0$.

5/ i) $p(x) = 1$, $q(x) = 0$, $w(x) = 1$, $\alpha_1 = 1$, $\alpha_2 = 0$

$\beta_1 = \beta_2 = 1$ so $q(x) > 0$, $\alpha_1, \alpha_2 > 0$, $\beta_1, \beta_2 > 0$

Now $q(x) = 0$ but $\alpha_1 = \beta_1 \neq 0$ so by

Theorem 4 the eigenvalues are positive.

ii) Soln has form $u(x) = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x$

$u(0) = 0 \Rightarrow A = 0$ $u'(x) = B \sqrt{\lambda} \cos \sqrt{\lambda} x$, $u(1) + u'(1) = 0$

or $B(\sin \sqrt{\lambda} + \sqrt{\lambda} \cos \sqrt{\lambda}) = 0$ $B \neq 0$

$\therefore \lambda$ satisfies given equation



Infinite set of eigenvalues at intersections of graphs.

5(cont) Eigenfunctions of form $\sin \sqrt{\lambda} x$

6/ i) $\frac{\partial^2 y}{\partial x^2} = 0.2 \frac{\pi}{L} \cos(\frac{\pi x}{L}) \cos(\frac{\pi ct}{L})$

$\frac{\partial^2 y}{\partial x^2} = -0.2 \frac{\pi^2}{L^2} \sin(\frac{\pi x}{L}) \cos(\frac{\pi ct}{L})$

$\frac{\partial^2 y}{\partial t^2} = -0.2 \frac{\pi^2 c^2}{L^2} \sin(\frac{\pi x}{L}) \sin(\frac{\pi ct}{L})$

$\frac{\partial^2 y}{\partial t^2} = -0.2 \frac{\pi^2 c^2}{L^2} \sin(\frac{\pi x}{L}) \cos(\frac{\pi ct}{L})$

$\therefore \frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} = 0$ and wave equation is satisfied.

ii) At $x = \frac{L}{2}$ $y = 0.2 \cos \frac{\pi ct}{L}$. Max value of $|\cos|$ is 1 so max amplitude = 20cm.

iii) Wavelength = $2L$ depth = $0.01 < 0.05$ wavelength \therefore shallow wave

$c^2 = gh = 100$ \therefore wave speed = 10 ms^{-1}

Period of oscillation = $\frac{L}{c} = 100 \text{ s}$

7/ i) $\text{curl } \underline{u} = (2x - x) \underline{i} + (y + 2z - (2x + y)) \underline{j} + (z - z) \underline{k}$
 $= 0$ \therefore irrotational

$\frac{\partial \phi}{\partial x} = yz + 2xz \Rightarrow \phi = yz^2 + x^2z + f(y, z)$

$\frac{\partial \phi}{\partial y} = xz \Rightarrow \frac{\partial f}{\partial y} = 0$ so $f(y, z) = g(z)$

$\frac{\partial \phi}{\partial z} = x^2 + xy \Rightarrow g'(z) = 0$ $\therefore g(z) = \text{const}$

$\therefore \phi(x, y, z) = x^2z + x^2z + C$

$\int_A^B \underline{u} \cdot d\underline{r} = \phi(B) - \phi(A) = -1$

At $t=0$ the circulation is zero by Stokes' Theorem as $\text{curl } \underline{u} = 0$.

To invoke Kelvin's theorem we need to know that the fluid is incompressible, inviscid and

moves under the action of conservative forces. Hence deduction

ii) $\nabla \psi = -b r \underline{e}_r - \frac{a}{r} \underline{e}_\theta$

$\underline{u} = \nabla \psi \times \underline{e}_z = br \underline{e}_\theta - \frac{a}{r} \underline{e}_r$

Superposition of sink of strength $2\pi a$ at origin and solid body rotation about origin

$\text{curl } \underline{u} = \frac{1}{r} \begin{vmatrix} \underline{e}_r & r \underline{e}_\theta & \underline{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ -\frac{a}{r} & br^2 & 0 \end{vmatrix} = 2b \underline{e}_z$

\therefore non-zero vorticity.