1997

3 i) a) $Q = uh = 2m^2s^{-1}$ $E = h + \frac{Q^2}{2h^2g} = 1{\cdot}2m$

For $Q = 2$, $E = 1{\cdot}2$, u and h must satisfy

$1{\cdot}2 = h + \frac{4}{20h^2}$ ie $5h^3 - 6h^2 + 1 = 0$

$(h-1)(5h^2 - h - 1) = 0$

$\therefore h = 1$ or $h = \frac{1 \pm \sqrt{21}}{10}$ So $h = 0{\cdot}56m$

(as -ve h impossible) giving $u = 3{\cdot}57ms^{-1}$

b) For $h=1$ $F_r = \frac{u}{\sqrt{gh}} = 0{\cdot}63 < 1$ deep & slow

For $h = 0{\cdot}56$ $F_r = 1{\cdot}51 > 1$ shallow and fast

ii) b) $\longleftarrow$ high speed
low pressure
Pressure force $\uparrow$
suction in passenger-to-be

4 i) Planes infinite, movement only in x-direction, no edge effects, no pressure gradients or body forces. 2d model with 2d flow

a) $u_2 = 0$ b) $\frac{\partial}{\partial y} = 0$ c) $\frac{\partial u}{\partial x} = 0$

Continuity $\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z} = 0 \underset{(c)}{\Rightarrow} \frac{\partial u_3}{\partial z} = 0$

boundary conditions $u_1 = Ue^{-t}$ $u_3 = 0$ on $z = a$

$u_1 = u_3 = 0$ on $z = 0$

$\frac{\partial u_3}{\partial z} = 0 \underset{(c)}{\Rightarrow} u_3 = $ Const but $u_3 = 0$ on $z = a$

so $u_3 = 0$ everywhere

$\therefore \underline{u} = u_1 \underline{i}$ (as and $u_3 = 0$)

From b) & c) $\underline{u} = u_1(z,t)\underline{i}$

ii) $\rho \frac{\partial u_1}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u_1}{\partial z^2}$ (1) $0 = -\frac{\partial p}{\partial z}$ (2)

(2) with (b) $\Rightarrow p = p(x,t)$

Sub $u_1 = e^{-t}f(z)$ $p = e^{-t}h(x)$ in (1)

$-\rho e^{-t} f(z) = -e^{-t}h'(x) + \mu e^{-t} f''(z)$

$h'(x) = \mu f''(z) + \rho f(z)$

LHS function of x only RHS function of z only

so $h'(x) = -C$ and $f''(z) + \frac{\rho}{\mu} f(z) = -\frac{C}{\mu}$ (3)

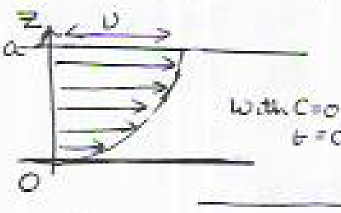
(3) gives $f(z) = A \cos\sqrt{\frac{\rho}{\mu}}z + B\sin\sqrt{\frac{\rho}{\mu}}z - \frac{C}{\rho}$

$u_1(z,t) = \left(A\cos\sqrt{\frac{\rho}{\mu}}z + B\sin\sqrt{\frac{\rho}{\mu}}z - \frac{C}{\rho}\right)e^{-t}$

When $z = 0$ $u_1 = 0 \Rightarrow A = \frac{C}{\rho}$

When $z = a$ $u_1 = Ue^{-t} \Rightarrow B = \frac{\frac{C}{\rho}\left(1 - \cos\sqrt{\frac{\rho}{\mu}}a\right) + U}{\sin\sqrt{\frac{\rho}{\mu}}a}$

iii)



With $C = 0$, $t = 0$ $\underline{u} = \frac{U\sin\sqrt{\frac{\rho}{\mu}}z}{\sin\sqrt{\frac{\rho}{\mu}}a}\,\underline{i}$

10 i) The standard form of $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$ where $A, \dots G$ are given functions of x and y is a simple form produced by an appropriate change of variable.

If the original equation is

elliptic, the form is $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \phi^2}$ + (terms of derivative order < 2).

hyperbolic, " " is $\frac{\partial^2 u}{\partial \xi \partial \phi} + \cdots$

parabolic " " " $\frac{\partial^2 u}{\partial \phi^2} + \cdots$ (ξ, ϕ new variables)

ii) 1/ wave, hyperbolic standard form $\frac{\partial^2 u}{\partial \xi \partial \phi} = 0$

2./ diffusion, parabolic - already in standard form

3/ Laplace, elliptic - " " " "

iii) Put $v = \frac{\partial u}{\partial \phi}$ (2) becomes $3\frac{\partial v}{\partial \xi} - v = 0$.

Integrating factor $e^{-\xi/3}$ giving $v = f(\phi)e^{\xi/3}$

$u = \int f(\phi)\,d\phi\, e^{\xi/3} + g(\xi) = e^{\xi/3}h(\phi) + g(\xi)$

$= e^{(y-x)/3}h(y - \frac{x}{4}) + g(y - x)$

$u(0,y) = e^{y/3}h(y) + g(y) = e^y$ (3)

$\frac{\partial u}{\partial x}(0,y) = -\frac{1}{3}e^{y/3}h(y) - \frac{1}{4}e^{y/3}h'(y) - g'(y) = e^y$ (4)

Diff (3) $\frac{1}{3}e^{y/3}h(y) + e^{y/3}h'(y) + g'(y) = e^y$ (5)

Add (4) & (5) gives $h'(y) = \frac{8}{3}e^{2y/3}$ $h(y) = 4e^{2y/3} + C$

From (3) $g(y) = -3e^y - Ce^{y/3}$

Sub in u gives $u = 4e^{y - x/2} - 3e^{y-x}$

11/ FS $\frac{1}{2}a_0 + \sum_{j=1}^{\infty} a_j \cos jx$ where $a_j = \frac{2}{\pi}\int_0^{\pi} \sin x \cos jx\,dx$

$a_j = -\frac{2}{\pi}\left[\frac{\cos(1-j)x}{2(1-j)} + \frac{\cos(1+j)x}{2(1+j)}\right]$ Hbk P6 $j \neq 1$

$= -\frac{1}{\pi}\left\{\left(\frac{(-1)^{j+1}}{1-j} + \frac{(-1)^{j+1}}{1+j}\right) - \left(\frac{1}{1-j} + \frac{1}{1+j}\right)\right\}$

$= -\frac{1}{\pi}\left\{\frac{2(-1)^{j+1}}{1-j^2} - \frac{2}{1-j^2}\right\} = \frac{2}{\pi}\left[\frac{1 + (-1)^j}{1-j^2}\right]$

$a_{2n+1} = 0$ $a_{2n} = \frac{4}{\pi(1-4n^2)}$

$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx = \frac{4}{\pi}$

$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\,dx = 0$. hence answer

ii) Theorem 4 Hbk P24 shows eigenvalues non-negative and 0 is eigenvalue, eigenfunction 1

For $\lambda > 0$ $X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$ A, B consts

$X' = -\sqrt{\lambda}A\sin\sqrt{\lambda}x + \sqrt{\lambda}B\cos\sqrt{\lambda}x$ $X'(0) \Rightarrow B = 0$

$X'(\pi) = 0 \Rightarrow \sin\sqrt{\lambda}\pi = 0$ so $\lambda = n^2$ $n = 1, 2, 3 \cdots$

hence answer

iii) a) $\frac{\partial u}{\partial x}(0,t) = 0$ $\frac{\partial u}{\partial x}(\pi, t) = 0$ $t > 0$

b) $u(x,t) = X(x)T(t)$ gives $T'(t) + k\lambda T(t) = 0$

and $X''(x) + \lambda X(x) = 0$ with $X'(0) = X'(\pi) = 0$

From ii) $\lambda_n = n^2$ $X_n(x) = \cos nx$ $n = 0, 1, 2, \cdots$

so $T_n(t) = e^{-n^2kt}$ and $u(x,t) = \sum_{n=0}^{\infty} a_n \cos nx\, e^{-n^2kt}$

$u(x,0) = \sin x$

so using i)

$\sum_{n=0}^{\infty} a_n \cos nx = \sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2 - 1}$

Comparing coefficients gives $a_0 = \frac{2}{\pi}$

$a_{2n} = -\frac{4}{\pi}\frac{1}{4n^2 - 1}$

$\therefore u(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx\, e^{-4n^2kt}}{4n^2 - 1}$.