Question 10

(i) State, *briefly*, what is meant by the *standard form* of a linear, second-order partial differential equation whose coefficients are, in general, functions of two independent variables. [2]

(ii) For each of the following three equations,

(a) state whether it is a wave equation, a diffusion equation or a Laplace equation,

(b) classify it, and

(c) if it is in standard form, state so; otherwise, give the second derivative terms of the standard form.

1. $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = 0$
2. $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} = 0$
3. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ [6]

(iii) The change of variables

$$\zeta = y - x, \quad \phi = y - \frac{1}{4}x$$

reduces the equation

$$4\frac{\partial^2 u}{\partial x^2} + 5\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{1}$$

to the form

$$3\frac{\partial^2 u}{\partial \zeta \partial \phi} - \frac{\partial u}{\partial \phi} = 0. \tag{2}$$

(You are *not* asked to derive Equation (2).)

Use Equation (2) to show that the general solution of Equation (1) is

$$u(x,y) = g(y-x) + e^{(y-x)/3}\, h\Big(y - \frac{x}{4}\Big),$$

where g and h are arbitrary functions. Hence determine the particular solution that satisfies the additional conditions

$$u(0,y) = e^y,$$
$$\frac{\partial u}{\partial x}(0,y) = e^y.$$ [12]