Question 4

The velocity vector field at time $t = 0$ for a fluid is

$$\mathbf{v} = (2xy + z^2)\mathbf{i} + (x^2 + y^2)\mathbf{j} + 2zx\mathbf{k}.$$

A closed path C is formed from the straight lines joining $O = (0, 0, 0)$, $A = (1, 0, 0)$, and $B = (1, 1, 0)$. These lines, in the $z = 0$ plane, are given as follows:

$$OA : y = 0 \text{ and } x = \alpha \quad (0 \le \alpha \le 1)$$
$$AB : x = 1 \text{ and } y = \beta \quad (0 \le \beta \le 1)$$

and $\quad OB : x = y = \gamma \quad (0 \le \gamma \le 1)$.

(i) By *evaluating* a line integral, show that the circulation of $\mathbf{v}$ around C is equal to zero.

(ii) Show further that the circulation is zero when C is *any* closed path in the fluid at time $t = 0$. [7]

Question 5

The function $u(x)$ satisfies the regular Sturm–Liouville problem

$$u''(x) + \lambda u(x) = 0 \quad (0 < x < 1),$$
$$u(0) = 0, \quad u(1) + u'(1) = 0.$$

(i) Explain carefully, but briefly, why each eigenvalue of this problem is positive.

(ii) Show that the eigenvalues satisfy the equation

$$\sqrt{\lambda} + \tan\sqrt{\lambda} = 0,$$

and by sketching two suitable functions on the same set of axes explain why the eigenvalues form an infinite set.

Give the form of the eigenfunctions of the problem. [7]

Question 6

As a result of two waves travelling in opposite directions along the length of a narrow lake, a standing wave is formed. The height, ζ, of the water surface with respect to some equilibrium position may be represented by the function

$$\zeta(x, t) = 0.2 \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi c t}{L}\right) \quad (0 \le x \le L, t \ge 0),$$

where L is the length of the lake, x is the horizontal distance along the lake from one end, t is the time and c is the wave speed of the travelling waves (all in SI units).

(i) Show that $\zeta(x, t)$ satisfies the wave equation.

(ii) What is the maximum vertical displacement (the amplitude) halfway along the lake, i.e. at $x = L/2$?

(iii) The length of the lake is 500 m and its equilibrium depth is 10 m. Use an appropriate approximation to find the wave speed when the wavelength is twice the length of the lake and find the period of oscillation of the wave. [You may take the magnitude of the acceleration due to gravity, g, as $10\,\mathrm{m\,s^{-2}}$.] [6]