1994 (Cont)

8 (iii) $\rho\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + u_3\frac{\partial u_1}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial z^2}\right) \Rightarrow \frac{\partial p}{\partial x} = \mu\frac{d^2 u_1}{dz^2}$ (1)

$\rho\left(\frac{\partial u_3}{\partial t} + u_1\frac{\partial u_3}{\partial x} + u_3\frac{\partial u_3}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial z^2}\right) \Rightarrow \frac{\partial p}{\partial z} = 0$ (2)

(2) $\Rightarrow p = p(x)$ as $\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$ & (1) $\Rightarrow \frac{dp}{dx} = \mu\frac{d^2u_1}{dz^2} = $ Const as LHS is a function of x & RHS is a function of z so $\frac{dp}{dx} = C$ and $p = Cx + p_0$

when $x = 0$ $p = p_0$ when $x = L$ $p = p_0 + \Delta p$ so $C = \frac{\Delta p}{L}$

$p = p_0 + \frac{\Delta p\, x}{L}$ and $\mu\frac{d^2u_1}{dz^2} = C = \frac{\Delta p}{L}$ ie $\frac{d^2u_1}{dz^2} = \frac{\Delta p}{\mu L}$

so $u_1 = \frac{\Delta p z^2}{\mu L 2} + Az + B$ $\quad u_1 = 0$ on $z = 0 \Rightarrow B = 0$

$u_1 = \frac{\omega D}{2}$ on $z = h \Rightarrow A = \frac{\omega D}{2h} - \frac{\Delta p h}{2\mu L}$

$\therefore u_1 = \frac{\Delta p}{L\mu}\frac{z^2}{2} + \frac{\omega D z}{2h} - \frac{\Delta p h z}{2\mu L}$ $\qquad Q = \int_0^h u_1\, dz$ gives one required

so $\Delta p = \left(\frac{\omega D h}{4} - Q\right)\frac{12L\mu}{h^3}$ which ≥ 0 for $Q \leq \frac{\omega D h}{4}$

9. i) Using Th 4 (Hbk P24) $q(x) = 0$ $\alpha_2 = 1$ $\alpha_1 = 0$ $\beta_1 = 1$ $\beta_2 = 0$ all eigenvalues are +ve.
For $\lambda > 0$ $X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. $X'(0) = 0 \Rightarrow B = 0$ $X(\pi) = 0 \Rightarrow A\cos\sqrt{\lambda}\pi = 0$
so $\sqrt{\lambda}\pi = \frac{(2n-1)\pi}{2}$ $n = 1, 2, \ldots$ $\lambda_n = \frac{(2n-1)^2}{4}$ eigenfunctions $X_n = \cos\left(\frac{2n-1}{2}\right)x$.

ii) a) $\cos 2x \cos\frac{x}{2} = \frac{1}{2}\left(\cos\left(\frac{5x}{2}\right) + \cos\left(\frac{3x}{2}\right)\right)$ (Hbk P11)

b) Let $u(x,t) = X(x)T(t)$ then $\frac{X''(x)}{X(x)} = \frac{T'(t) + k\gamma^2 T(t)}{kT(t)} = -\lambda$

ie $X'' + \lambda X = 0$ (1)

and $T' + k(\gamma^2 + \lambda)T = 0$ (2) $\quad$ bc's $X'(0) = X(\pi) = 0$ (3)

(1) & (3) $\Rightarrow X_n = \cos\left(\frac{2n-1}{2}\right)x$ $n = 1, 2, 3 \ldots$ from part (i)

Corresponding $T_n = b_n \exp\left\{-k\left(\gamma^2 + \frac{(2n-1)^2}{4}\right)t\right\}$

Using superposition $u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left\{-k\left(\gamma^2 + \frac{(2n-1)^2}{4}\right)t\right\}\cos\left(\frac{(2n-1)x}{2}\right)$

Initial condition gives

$$\sum_{n=1}^{\infty} b_n \cos\left(\frac{2n-1}{2}\right)x = \frac{1}{2}\left[\cos\left(\frac{3x}{2}\right) + \cos\left(\frac{5x}{2}\right)\right]$$ equating coeffs gives

$b_1 = 0$, $b_2 = \frac{1}{2}$ $b_3 = \frac{1}{2}$ $b_n = 0\ (n \geq 4)$ so

$u(x,t) = \frac{1}{2}e^{-k(\gamma^2 + 9/4)t}\cos\left(\frac{3x}{2}\right) + \frac{1}{2}e^{-k(\gamma^2 + \frac{25}{4})t}\cos\left(\frac{5x}{2}\right)$.

10. i) a) $u(x,t) = X(x)T(t)$ then $\frac{\partial}{\partial x}\left(\frac{1}{A}(AX)'T\right) - \frac{1}{c^2}XT'' = 0$, $\frac{1}{X}\left(\frac{(AX)'}{A}\right)' = \frac{T''}{c^2 T} = -\lambda$

$\left(\frac{A'X}{A} + X'\right)' + \lambda X = 0$

or $X'' + \frac{A'}{A}X' + \left(\frac{A'}{A}\right)'X + \lambda X = 0$ So $f = \frac{A'}{A}$

bc's $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \Rightarrow X'(0) = X'(L) = 0$

b) when $A(x)$ is constant $f' = f = 0$ so $X'' + \lambda X = 0$ $X'(0) = X'(L) = 0$

By Theorem 4, eigenvalues are positive or zero so ... $\lambda_n = n^2\pi^2/L^2$ with $X_n = \cos\left(\frac{n\pi x}{L}\right)$ and $\lambda = 0$ with $X_0 = 1$.

ii) if $A(x) = $ const $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$. $p(x) = \cos x$ $q(x) = \sin x$

d'Alembert's solution gives

$$u(x,t) = \frac{1}{2}\left[\cos(x - ct) + \cos(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\sin s\, ds$$

$$= \frac{1}{2}\left(\cos(x - ct) + \cos(x + ct)\right) - \frac{1}{2c}\left(\cos(x + ct) - \cos(x - ct)\right)$$

This is the sum of standing waves but can be written as

$u(x,t) = \cos x \cos ct + \frac{1}{c}\sin x \sin ct$ so $u(x,0) = \cos x$ $\frac{\partial u}{\partial t} = -c\cos x \sin ct + \sin x \cos ct$

so $\frac{\partial u}{\partial t}(x,0) = \sin x$

D'Alembert's Soln applies to an unbounded medium. For finite length we assume conditions apply only over that length, and define even extensions about $x = 0$ and $x = L$. Then d'Alembert's soln could be used.