Question 10

Plane sound waves propagate in a musical instrument (a pipe) of length ℓ. This situation can be modelled by the equation

$$\frac{\partial}{\partial x}\left[\frac{1}{A(x)}\frac{\partial}{\partial x}\left(A(x)u(x,t)\right)\right] - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(x,t) = 0 \quad (0 < x < \ell,\ A(x) \neq 0), \qquad (1)$$

where $u(x,t)$ is the position at time t of an air particle whose equilibrium position is x, measured from one end of the instrument, $A(x)$ is the variable cross-sectional area of the pipe, and c is the speed of sound. The two ends of the instrument are open to the air so that there are free-end boundary conditions on u at these two ends.

(i) (a) Use the method of separation of variables to show, clearly, that there are solutions of Equation (1) of the form

$$u(x,t) = X(x)T(t)$$

for which $X(x)$ satisfies the Sturm–Liouville problem

$$\left.\begin{array}{l} X''(x) + f(x)X'(x) + (f'(x) + \lambda)X(x) = 0, \\ X'(0) = X'(\ell) = 0. \end{array}\right\} \qquad (2)$$

Give the expression for $f(x)$ in terms of $A(x)$.

(b) Find the eigenvalues and eigenfunctions of the problem given by Equations (2) in the case where $A(x)$ is a non-zero constant. [12]

(ii) Apply d'Alembert's solution to the problem of sound waves travelling in an infinitely long pipe of constant cross-section, that is to solving Equation (1) in the case where $A(x)$ is constant, when u is subject to the initial conditions:

$$\left.\begin{array}{ll} u(x,0) = \cos x & (-\infty < x < \infty), \\ \dfrac{\partial u}{\partial t}(x,0) = \sin x & (-\infty < x < \infty). \end{array}\right\} \qquad (3)$$

Show that the solution to this problem can be written as a sum of standing waves and confirm that it satisfies the initial conditions (3). Explain, *briefly*, how you would tackle the problem of finding $u(x,t)$ for waves travelling in a pipe (musical instrument) of finite length using a method involving d'Alembert's solution. [8]

$\ddot{x} + \omega^2 x = 0$

Question 11

(i) Explain *briefly* what you understand by the following:

(a) *a fluid*, *shear stress* and *compressibility*, using your explanations to distinguish between solids, liquids and gases;

(b) *the Magnus effect*, giving at least two examples of where this may be observed or used in everyday life;

(c) *d'Alembert's paradox* and how it is resolved;

(d) *the incompressibility of a fluid and of a flow*, including a brief explanation of the nature of the volume flow rate of a liquid across a surface S enclosing a volume V. [15]

(ii) Sketch a graph of the variation of c^2 with h (≥ 0) where c is the speed of waves travelling in water of depth h. Use your graph to explain the refraction of water waves as they approach a sloping beach. [5]

[END OF QUESTION PAPER]