Question 9

(i) Show that the eigenvalue problem

$$X''(x) + \lambda X(x) = 0 \quad (0 < x < \pi),$$
$$X'(0) = X(\pi) = 0,$$

has eigenvalues $\lambda_n = (2n-1)^2/4$ and corresponding eigenfunctions

$$X_n(x) = \cos\left(\frac{2n-1}{2}\right)x \quad (n = 1, 2, \ldots). \qquad [5]$$

(ii) A porous rod of length π containing moisture has one end sealed and the other end in contact with a dry medium, and loses moisture through its surface to dry air. The concentration of moisture $u(x,t)$ satisfies the problem given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} + \gamma^2 u \quad (0 < x < \pi, t > 0),$$

with

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad u(\pi,t) = 0 \quad (t \geq 0)$$

and

$$u(x,0) = \cos 2x \cos\frac{x}{2} \quad (0 < x < \pi),$$

where x and t represent distance along the rod and time, respectively, and k (> 0) and γ are constants.

(a) Use a trigonometric identity to show that the given initial condition can be expressed as a sum of cosine terms.

(b) Use the method of separation of variables and the result of Part (i) to determine the concentration of moisture in the rod at times subsequent to $t = 0$. [15]