1994

1/ $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \cos t \frac{dy}{dx} = \sqrt{1-\sin^2 t}\,\frac{dy}{dx}$

$= \sqrt{1-x^2}\,\frac{dy}{dx}$

$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \sqrt{1-x^2}\,\frac{d}{dx}\left(\sqrt{1-x^2}\,\frac{dy}{dx}\right)$

$= -x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2}$

equation becomes $\frac{d^2y}{dt^2} + 4y = 0 \quad 0 < t < \frac{\pi}{2}$

soln $y = A\cos(2t) + B\sin(2t)$

so $y = A\cos(2\arcsin x) + B\sin(2\arcsin x)$

$x/y = A(1-2x^2) + Bx\sqrt{1-x^2}$

2/ i) $\frac{\partial u}{\partial x} + \frac{y}{x}\frac{\partial u}{\partial y} - \frac{u}{x} = 0$ so $\frac{dy}{dx} = \frac{y}{x}$

ie $\frac{y}{x} = c$ characteristic curves $y = cx$

ii) Slopes of characteristics given by

$x^2 s^2 - y^2 = 0 \quad s = \pm y/x$ so $y = c_1 x$

$xy = c_2$.

3. i) Assume $\frac{\Delta p}{L} = k d^{\alpha}\rho^{\beta} V_m^{\gamma}\mu^{\delta}$

using dimensions

$ML^{-1}T^{-2}L^{-1} = L^{\alpha}M^{\beta}L^{-3\beta}L^{\gamma}T^{-\gamma}M^{\delta}L^{-\delta}T^{-\delta}$

giving $1 = \beta + \delta$, $-2 = \alpha - 3\beta + \gamma - \delta$

$-2 = -\gamma - \delta$ let $\delta = a$ then $\beta = 1-a$

$\gamma = 2 - a$, $\alpha = a - 1$. Hence answer.

ii) $Q = \frac{\pi d^4 C}{8\mu\, 16}$ Hbk P24 with $a = \frac{d}{2}$

$C = \frac{\Delta p}{L} \quad Q = \frac{\pi d^2 V_m}{4}$ (by continuity)

Sub into above gives $\frac{\Delta p}{L} = \frac{V_m 32\mu}{d^2}$

$\therefore f(Re) = \frac{32}{Re}$ $\quad = \frac{\rho V_m^2}{d}\left(\frac{32\mu}{\rho V_m d}\right)$

4 i) Continuity $\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho u) = 0$

No θ, ϕ dependence so in spherical polar $\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho u_r) = 0$

differentiating product gives result

ii) ρ = const so $\frac{\partial}{\partial r}(r^2 u_r) = 0$

ie $r^2 u_r = A$ at $r = a$ $u_r = U$ so $A = a^2 U$

$u_r = \frac{dr}{dt} = \frac{a^2 U}{r^2}$ integration gives $\frac{r^3}{3} = a^2 U t + C$

$r = a$ when $t = 0$ so $C = \frac{a^3}{3}$. Hence ans

5. By continuity $0.09 v_1 = 0.12 v_2$ (width const)

v_1 = upstream speed v_2 = downstream speed

so $v_1 = 4/3\, v_2$. Using Bernoulli's eqn along a surface streamline, using base of channel upstream as datum

$\frac{p_0}{\rho} + \frac{1}{2}v_1^2 + 0.09g = \frac{p_0}{\rho} + \frac{1}{2}v_2^2 + 0.18g$

$\therefore \frac{16}{9}v_2 - v_2^2 = 0.18g$ (cross sectional area)

or $v_2 = 1.5212$

Volume flow rate $= 0.36 \times 1.52$

$= 0.55\ m^3 s^{-1}$

b 2d p

6. i) $\int_{OA} \underline{v}\cdot d\underline{r} = \int_0^1 \underline{v}\cdot\frac{d\underline{r}}{dx}\,dx = \int_0^1 x^2\,dx = \frac{1}{3}$

$\int_{AB} \underline{v}\cdot d\underline{r}\ \int_0^1 2s\,ds = 1 \quad \int_{BO} \underline{v}\cdot d\underline{r} = \int_1^0 4\gamma^2 d\gamma = -\frac{4}{3}$

$\therefore \oint_C \underline{v}\cdot d\underline{r} = \frac{1}{3} + 1 - 4/3 = 0$

ii) curl $\underline{v} = \underline{i}(0-0) + \underline{j}(0-0) + \underline{k}(2y-2y) = 0$

From Stokes' Theorem, circulation around all curves is zero (Hbk p18 & 21).

7. i) $\underline{u} = -2y\underline{i} + 2xt\underline{j}$

$\nabla\cdot\underline{u} = 0$ — incompressible

$\nabla\times\underline{u} = 0\underline{i} + 0\underline{j} + (2t+2)\underline{k} \neq \underline{0}$ in general

— not irrotational $\frac{\partial u}{\partial t} = 2x\underline{j} \neq \underline{0}$ for all x

so flow is not steady.

ii) $\frac{\partial\psi}{\partial y} = -2y$ ①, $\frac{\partial\psi}{\partial x} = -2xt$ ②

Integrate ① $\psi = -y^2 + f(x,t)$

so $\frac{\partial\psi}{\partial x} = \frac{\partial f}{\partial x} = -2xt$ from ② $f = -x^2 t + g(t)$

$\psi = -y^2 - x^2 t + g(t)$. Streamlines at $t = 1$

$y^2 + x^2 = $ const

iii) Euler's equation

$\rho\left(\frac{\partial\underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u}\right) = -\nabla p + \rho\underline{F}$

$\rho\left(2x\underline{j} + \left(-2y\frac{\partial}{\partial x} + 2xt\frac{\partial}{\partial y}\right)(-2y\underline{i} + 2xt\underline{j})\right)$

$= -\nabla p + \rho 2x\underline{j}$

$\underline{i}$ comp gives $\frac{\partial p}{\partial x} = 4\rho xt$ ①

$\underline{j}$ comp gives $\frac{\partial p}{\partial y} = 4\rho yt$ ②

$\underline{k}$ comp gives $\frac{\partial p}{\partial z} = 0$ ③

③ implies $p = p(x, y, t)$

② $\Rightarrow$ $p = 2\rho y^2 t + f(x,t)$

① $\Rightarrow \frac{\partial f}{\partial x} = 4\rho xt \Rightarrow f(x,t) = 2\rho x^2 t + g(t)$

so $p = 2\rho t(x^2+y^2) + g(t)$

At $t = 1$ along streamline $x^2 + y^2$ = const

so $p = 2\rho \times$ const + const $\Rightarrow p$ = const.

8. i) $u_2 = 0$ everywhere $\frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y} = 0$ - no y-dependence.

on $z = 0$ $u_1 = u_3 = 0$ (AB)

on $z = h$ $u_1 = \frac{\omega D}{2}$ $u_3 = 0$ (A'B')

ii) The drum is rotating at constant angular speed.

steady $\frac{\partial}{\partial t} = 0$ — no time dependence.

no forces in z-direction, $u_3 = 0$ on both boundaries so $u_3 = 0$.

Fluid is liquid so incompressibility reasonable

$\nabla\cdot\underline{u} = 0$ ie $\frac{\partial u_1}{\partial x} = 0$ as $u_3 = 0$

$\frac{\partial u_1}{\partial y} = \frac{\partial u_1}{\partial x} = \frac{\partial u_1}{\partial t} = 0$ so $u_1 = u_1(z)$