Question 8

A viscosity pump is made up of a stationary, cylindrical casing and a drum of diameter D that rotates inside the casing at a constant angular speed ω; the drum and the casing are concentric. Liquid with density ρ and coefficient of viscosity μ enters a section AA′, flows around the annulus between the drum and casing, and leaves at another section BB′ (see Figure 1a). The pressure at BB′ is higher than at AA′, by Δp. The gap of the annulus h is so small compared with the diameter of the drum that the flow can be considered as that between two flat, parallel plates of length L, as shown in Figure 1b.

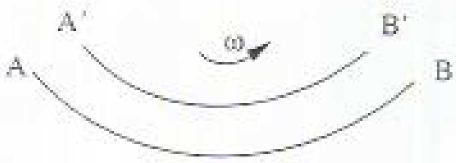

Figure 1a

Figure 1b

Choose a Cartesian coordinate system so that the flow is in the x-direction and the lower plate AB lies in the plane $z = 0$.

(i) Assume that the flow is laminar and two-dimensional. What can you say about the y-component of the velocity vector $\mathbf{u} = (u_1, u_2, u_3)$, and the y-dependence of the other velocity components, u_1 and u_3?

Write down the boundary conditions that the velocity vector $\mathbf{u}$ must satisfy on the plates AB and A′B′. *[3]*

(ii) Assume that the flow is fully developed and that there are no body forces. Taken together with these assumptions, what else in the question statement suggests that the flow is steady? Write down the mathematical implications of steady flow.

Explain why it is reasonable to take $u_3 = 0$ everywhere.

Why is it reasonable to assume that the fluid is incompressible? Write down a mathematical statement of incompressibility involving $\mathbf{u}$. Hence show that $u_1 = u_1(z)$. *[5]*

(iii) Write down the x- and z- components of the Navier–Stokes equations for this flow. Solve them together with the boundary conditions of Part (i) to find the pressure p and velocity component u_1, and hence show that the flow rate (per unit depth in the y-direction) Q is given by

$$Q = \frac{\omega D h}{4} - \frac{\Delta p h^3}{12 L \mu}.$$

Deduce that Q is less than $\omega D h/4$. *[12]*