Part I

The questions in this part are not all worth the same number of marks.

The number of marks assigned to each question is given in square brackets.

Part I as a whole carries 40% of the total examination marks.

Question 1

Use the change of variable $x = \sin t$ $(0 < t < \pi/2)$ to find the general solution, in terms of x, of the differential equation

$$(1 - x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} + 4y = 0 \qquad (0 < x < 1).$$

[7]

Question 2

Find a set of characteristic curves for each of the following differential equations, in which u is a function of x and y:

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} - u = 0 \qquad (x \neq 0)$;

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} - y^2\dfrac{\partial^2 u}{\partial y^2} - u = 0 \qquad (x \neq 0, y \neq 0)$.

[6]

Question 3

(i) A fluid of density ρ and coefficient of viscosity μ flows along a pipe of diameter d with a mean speed V_m. A constant pressure difference Δp is applied along a length L of the pipe. Use the method of dimensional analysis to show that the pressure gradient is given by

$$\frac{\Delta p}{L} = \frac{\rho V_m^2}{d} f(Re),$$

where f is an undetermined function of the Reynolds number $Re = \rho d V_m / \mu$.

(ii) Rearrange the Hagen–Poiseuille formula to find the specific form for f when the pressure gradient is constant.

[7]

Question 4

A fluid flows radially in all directions from a point with a spherically symmetric velocity

$$\mathbf{u} = u_r(r, t)\mathbf{e}_r,$$

where r and $\mathbf{e}_r$ are respectively the radial distance and unit radial vector from the point and t is time.

(i) *Show* that the continuity equation for the fluid, except at the point itself, can be expressed in the form

$$\frac{\partial \rho}{\partial t} + \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + u_r\frac{\partial \rho}{\partial r} = 0 \qquad (r > 0),$$

where ρ is the density of the fluid.

(ii) Suppose now that the fluid is incompressible and the flow is steady. Find an expression for u_r in terms of r and show that the path of a particle that lies on the sphere $r = a$ and has speed $u_r = U$, at $t = 0$, is given by

$$r^3 = 3Ua^2 t + a^3.$$

[7]