Question 4

Find the solution $u(r,\theta)$ of Laplace's equation in spherical polar coordinates with cylindrical symmetry, namely

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0 \qquad (0\le r<R,\ 0\le\theta\le\pi),$$

that satisfies the conditions

$$u(r,\theta) \text{ is bounded and } u(R,\theta)=2\cos^2\frac{\theta}{2}.$$

(The general solution to Laplace's equation in spherical polar coordinates which is periodic in θ may be assumed.) [6]

Question 5

Find, correct to four decimal places, $u(\frac{1}{2}\pi,\frac{1}{2}\pi)$ where $u(x,t)$ satisfies

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2} \qquad (0<x<\infty,\ t>0)$$

subject to the initial conditions

$$u(x,0)=0 \quad (0<x<\infty),$$
$$\frac{\partial u}{\partial t}(x,0)=e^{-x}\cos x \quad (0<x<\infty),$$

and when there is a free-end boundary condition at $x=0$. [6]

Question 6

The 'wave condition' for waves in an incompressible inviscid fluid of infinite depth for which surface tension effects are considered is

$$\omega^2=gk+\frac{Tk^3}{\rho}$$

where T is the magnitude of the surface tension force per unit length, ρ is the density and ω, g and k have their usual meanings.

(i) Using the above wave condition, express the wave speed, c, in terms of the wavelength, λ, and the constants T and ρ.

(ii) Show that $S=Tk^2/(\rho g)$ is a dimensionless parameter.

(iii) Find an expression for the group velocity, c_g, of these waves and hence show that

$$c_g=\frac{g^{1/2}(1+3S)}{2k^{1/2}(1+S)^{1/2}} \quad \text{and} \quad c=\left(\frac{g}{k}\right)^{1/2}(1+S)^{1/2}.$$

(iv) Hence show that when surface tension effects are negligible, so that $S\ll 1$, then the waves are the familiar deep water waves for which $c_g\simeq c/2$. [7]