Part I

Answer **ALL SIX** *questions in this part.*

The questions in this part are not all worth the same number of marks.

The number of marks assigned to each question is given in square brackets.

Part I as a whole carries 40% of the total examination marks.

Question 1

Solve the following problems:

(i) $u'(x) + u(x) = 0, \quad u(0) = 1;$

(ii) $u''(x) + u(x) = 0, \quad u(0) = 1, \quad u(\pi/2) = 0;$

(iii) $x^2 u''(x) + xu'(x) + u(x) = 0 \quad (x > 0), \quad u(1) = 0, \quad u(e) = \sin 1.$ [7]

Question 2

A tornado is modelled as follows. It has a core of radius a rotating at a constant angular frequency ω. Outside the core the motion is that of a vortex of constant strength $2\pi k$, so that the velocity field modelling the tornado, in cylindrical polar coordinates with the vertical axis of the tornado as the z-axis, is

$$\mathbf{u} = \begin{cases} \omega r \mathbf{e}_\theta, & r < a, \\ \dfrac{k}{r}\mathbf{e}_\theta, & r \geq a. \end{cases}$$

(i) Show that this velocity field satisfies the continuity equation for incompressible flow.

(ii) Write down the relation between ω and k that must be satisfied for $\mathbf{u}$ to be everywhere continuous.

(iii) Let $\Gamma = \Gamma(r)$ be the circulation around circles of radius r in planes perpendicular to the z-axis and with centres on that axis.

(a) Show that *outside* the core the motion is irrotational and that the circulation, Γ, is non-zero and constant.

(b) Show also, by finding expressions for the vorticity and for the circulation inside the core, that *inside* the core the vorticity and circulation, Γ, are both non-zero (for $r \neq 0$). [7]

Question 3

(i) Show that, for any incompressible flow, the volume flow rate across a streamline is zero.

(ii) An incompressible, inviscid fluid flows steadily with uniform speed u along a horizontal channel of uniform semi-circular cross-section of radius r.

(a) Express the volume flow rate, V, in terms of u and r, when the fluid fills the channel.

(b) For a given V, show that the relation between the critical speed u_c and the critical depth h_c is given by

$$u_c = \left[gA(h_c) \Big/ \frac{dA}{dh}(h_c) \right]^{\frac{1}{2}}$$

where A is the cross-sectional area of fluid in the channel of depth $h \leq r$. [7]