Question 9

(i) Find the general solution $w = w(x, y)$ of the partial differential equation

$$\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x} = 0.$$ [3]

(ii) The function $u(x, y)$ satisfies the partial differential equation

$$3x^2 \frac{\partial^2 u}{\partial y^2} - \frac{1}{12}\frac{\partial^2 u}{\partial x^2} - 6x^2 \frac{\partial u}{\partial y} + \left(x + \frac{1}{12x}\right)\frac{\partial u}{\partial x} = 0 \quad (x \neq 0). \qquad (1)$$

(a) Show that this equation is hyperbolic in the region R of the (x, y)-plane over which it is defined.

(b) Find the equations of the characteristic curves in the region R and hence show that the characteristic coordinates may be chosen to be

$$\zeta = y - 3x^2, \quad \phi = y + 3x^2.$$

(c) Use these characteristic coordinates and the chain rule to transform the partial differential Equation (1) to its standard form.

(d) Hence, and using the result of Part (i), above, find the general solution $u = u(x, y)$ of Equation (1). [17]

Question 10

(i) Show that the Fourier cosine series for the function

$$f(x) = x + \pi$$

on the interval $0 < x < \pi$ is

$$\frac{3\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos(2m-1)x}{(2m-1)^2}.$$ [7]

(ii) Show that the eigenvalue problem

$$X''(x) + \lambda X(x) = 0 \qquad (0 < x < \pi)$$
$$X'(0) = X'(\pi) = 0$$

has eigenvalues $\lambda_n = n^2 \quad (n = 0, 1, 2, \ldots)$ and corresponding eigenfunctions $X_n(x) = \cos nx$. [4]

(iii) The equation governing the temperature distribution $u(x, t)$ in an insulated bar of length π is given by the diffusion equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad (0 < x < \pi,\ t > 0),$$

where x and t represent distance along the bar and time respectively and k is a positive constant. Initially the temperature distribution of the bar is

$$u(x, 0) = x + \pi \quad (0 < x < \pi).$$

The two ends of the bar are insulated (so that there is no temperature change across the ends).

(a) Write down the boundary conditions at $x = 0$ and $x = \pi$ for the temperature distribution.

(b) Use the method of separation of variables to determine the temperature distribution $u(x, t)$ at times $t > 0$. [9]