(ii) Now $c = \frac{\omega}{k}$ & $\lambda = \frac{2\pi}{k}$

$$\therefore \frac{4\pi^2 c^2}{\lambda^2} = g \frac{2\pi}{\lambda} \tanh \frac{2\pi}{\lambda} h$$

or $$c^2 = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}.$$

PART II

7. (i) Boundaries and boundary conditions do not depend on x & y, so we would not expect $\underset{\sim}{u}$ to depend on x & y.
As boundaries are parallel to xy-plane and boundary conditions have no y-component, we would not expect $\underset{\sim}{u}$ to have any y-component.

(ii) $$\underset{\sim}{u}(x,y,0,t) = \underset{\sim}{0}$$
$$\underset{\sim}{u}(x,y,h,t) = Ue^{-\alpha t}\underset{\sim}{i}$$

(iii) From continuity equation
$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = 0$$
Using the modelling assumptions this reduces to $\frac{\partial u_3}{\partial z} = 0$ which gives $u_3 = f(t)$

From b.c.'s $u_3 = 0$

From modelling assumptions $u_2 = 0$

Hence $\underset{\sim}{u}(x,y,z,t) = u_1(z,t)\underset{\sim}{i}$

(iv) In Navier Stokes equation
$$\nabla p = \underset{\sim}{0} \quad \& \quad \underset{\sim}{F} = \underset{\sim}{0}$$

So $$\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u}.\underset{\sim}{\nabla})\underset{\sim}{u}\right) = \mu \nabla^2 \underset{\sim}{u}$$

As $\underset{\sim}{u} = u_1(z,t)\underset{\sim}{i}$, we have $\frac{\partial \underset{\sim}{u}}{\partial t} = \frac{\partial u_1}{\partial t}\underset{\sim}{i}$,

$$(\underset{\sim}{u}.\underset{\sim}{\nabla})\underset{\sim}{u} = \left(u_1\frac{\partial}{\partial x}\right)u_1(z,t)\underset{\sim}{i} = \underset{\sim}{0}, \quad \nabla^2\underset{\sim}{u} = \left(\frac{\partial^2}{\partial z^2}\right)u_1(z,t)\underset{\sim}{i}$$

Hence $$\rho\frac{\partial u_1}{\partial t} = \mu\frac{\partial^2 u_1}{\partial z^2}.$$

(v) Assuming $u_1(z,t) = e^{-\alpha t}f(z)$ the P.D.E. becomes $$f''(z) + \frac{\alpha\rho}{\mu}f(z) = 0.$$

So $$f(z) = A\cos\sqrt{\frac{\alpha\rho}{\mu}}\,z + B\sin\sqrt{\frac{\alpha\rho}{\mu}}\,z$$

Now $f(0) = 0$ $\quad\therefore A = 0$

& $f(h) = U$ $\quad\therefore B = \frac{U}{\sin\sqrt{\frac{\alpha\rho}{\mu}}\,h}$

$$\therefore \quad u_1(z,t) = U\frac{\sin\sqrt{\frac{\alpha\rho}{\mu}}\,z}{\sin\sqrt{\frac{\alpha\rho}{\mu}}\,h}e^{-\alpha t}$$

(vi) The force per unit area on $z = 0$ is
$$\mu\frac{\partial u_1}{\partial z}\Big|_{z=0} = \mu U\sqrt{\frac{\alpha\rho}{\mu}}\operatorname{cosec}\sqrt{\frac{\alpha\rho}{\mu}}\,h\;e^{-\alpha t}.$$

8(i) $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad (0<x<\pi)$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi}\cos x\sin nx\,dx$$