$$= \frac{1}{\pi}\int_0^{\pi}(\sin(n+1)x + \sin(n-1)x)\,dx$$

$$= -\frac{1}{\pi}\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \qquad (n \neq 1)$$

$$= \frac{1}{\pi}\left\{\left(\frac{1}{n+1} + \frac{1}{n-1}\right) - \left(\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right)\right\}$$

$$= \frac{2n}{(n^2-1)\pi}(1+\cos n\pi)$$

$$= \begin{cases} \dfrac{4n}{(n^2-1)\pi} & n = 2,4,6,\ldots \\ 0 & n = 3,5,7,\ldots \end{cases}$$

For $n=1$, $\quad b_1 = \frac{2}{\pi}\int_0^{\pi}\cos x \sin x\,dx$

$$= \frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx = 0$$

Hence $\quad f(x) = \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{m \sin 2mx}{4m^2-1}$

(ii) $\quad X'' + \lambda X = 0 \qquad (0<x<\pi)$

$X(0) = 0$ & $X(\pi) = 0$.

$\lambda \le 0$ leads to trivial solutions.

If $\lambda > 0$ $\quad X = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$

$\left.\begin{matrix} X(0)=0 \\ X(\pi)=0 \end{matrix}\right\}$ $\quad A = 0$, $\quad \sin\sqrt{\lambda}\,\pi = 0$

i.e. $\sqrt{\lambda} = n \qquad n = 1,2,3,\ldots$

So $\quad \lambda_n = n^2 \qquad n = 1,2,3,\ldots$

and $\quad X_n(x) = B_n \sin nx$

(iii) (a) $\quad u(0,t) = 0 \qquad t>0$

$u(\pi,t) = 0 \qquad t>0$

(b) Using separation of variables $u(x,t) = X(x)T(t)$ we obtain

$\left.\begin{matrix} X''(x) + \lambda X(x) = 0 \\ \text{with} \quad X(0) = 0 \ \& \ X(\pi) = 0 \end{matrix}\right\}$

and $\quad T'(t) + k\lambda T(t) = 0$.

The eigenvalue problem gives

$\lambda_n = n^2, \quad X_n(x) = \sin nx \qquad n = 1,2,3,\ldots$

So $\quad T_n(t) = e^{-n^2kt}$

& $\quad u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx\, e^{-n^2kt}$.

$u(x,0) = \cos x\} \therefore \sum_{n=1}^{\infty} b_n \sin nx = \cos x$

$$= \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{m\sin 2mx}{4m^2-1}$$

So $\quad b_n = \begin{cases} 0 & n = 1,3,5,\ldots \\ \dfrac{4}{\pi}\dfrac{n}{n^2-1} & n = 2,4,6,\ldots \end{cases}$

& $\quad u(x,t) = \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{m}{4m^2-1}\sin 2mx\, e^{-4m^2kt}$.

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9. (i) As the Sturm-Liouville problem is regular with $w(x) = 1$,

$$\int_0^1 u_m(x)u_n(x)\,dx = 0$$

where u_m, u_n are eigenfunctions corresponding to distinct eigenvalues.

(ii) Now $\quad p_{min} = 1, \quad p_{max} = 25,$

$q_{min} = 1, \quad q_{max} = 1,$

$w_{min} = 1, \quad w_{max} = 1.$

So by Theorem 5 (HB pg 29), $\sigma_n \le \lambda_n \le \tau_n$, where σ_n are the eigenvalues of

$\left.\begin{matrix} y'' + (\sigma-1)y = 0, \\ y(0) = 0, \ y(1) = 0 \end{matrix}\right\}$

and τ_n are the eigenvalues of

$\left.\begin{matrix} y'' + \frac{1}{25}(\tau-1)y = 0, \\ y(0) = 0, \ y(1) = 0. \end{matrix}\right\}$