Question 6

The following equations are used in the linear theory of gravity water waves when the water is of finite equilibrium depth h:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0 \quad (0<z<h),$$
$$\frac{\partial\phi}{\partial z}=0 \qquad \text{at } z=0,$$
$$\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial z}=0 \quad \text{at } z=h,$$

where ϕ is the velocity potential and g is the magnitude of the acceleration due to gravity. In these equations the channel floor is $z=0$ and the undisturbed free surface is $z=h$ with z measured vertically upwards.

By considering a solution of the form

$$\phi(x,z,t)=f(z)\cos(kx-\omega t),$$

show that

(i) the frequency ω is related to the wave number k by the dispersion relation

$$\omega^2=gk\tanh kh;$$

(ii) the relationship between the wave speed c and wavelength λ is

$$c^2=\frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda}.$$

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