Now $\sigma_n - 1 = n^2\pi^2$ i.e. $\sigma_n = n^2\pi^2+1$

and $\frac{1}{25}(\tau_n - 1) = n^2\pi^2$ i.e. $\tau_n = 25n^2\pi^2+1$.

So $n^2\pi^2+1 \leq \lambda_n \leq 25n^2\pi^2+1$.

The smallest eigenvalue is λ_1 and so

$$\lambda_n \geqslant \lambda_1 \geqslant \pi^2+1 > 5.$$

(iii) $x = \frac{1}{4}(t-1)$ & $t = 4x+1$

So $\frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx} = 4\frac{du}{dt}$ & $\frac{d}{dx} = 4\frac{d}{dt}$

In DE $\}$ $4\frac{d}{dt}\left(t^2\left(4\frac{du}{dt}\right)\right) - u + \lambda u = 0$

i.e. $16t^2\frac{d^2u}{dt^2} + 32t\frac{du}{dt} + (\lambda-1)u = 0$.

This is a Cauchy - Euler equation with indicial equation

$$16\alpha(\alpha-1) + 32\alpha + (\lambda-1) = 0$$

or $16\alpha^2 + 16\alpha + (\lambda-1) = 0$

$\therefore$ $$\alpha = \frac{-16 \pm \sqrt{256-64(\lambda-1)}}{32}$$

$$= -\tfrac{1}{2} \pm \tfrac{1}{4} i\sqrt{\lambda-5}.$$

Since $\lambda > 5$, the solution of the D.E. is

$$u = t^{-1/2}\left\{A\cos\left(\tfrac{1}{4}\omega\log_e t\right) + B\sin\left(\tfrac{1}{4}\omega\log_e t\right)\right\}$$

where $\omega^2 = \lambda - 5$

$$= (4x+1)^{-\frac{1}{2}}\left\{A\cos\left(\tfrac{1}{4}\omega\log_e(4x+1)\right) + B\sin\left(\tfrac{1}{4}\omega\log_e(4x+1)\right)\right\}$$

Now $u(0) = 0\}$ $\therefore$ $A = 0$

$u(1) = 0\}$ $\sin\left(\tfrac{1}{4}\omega\log_e 5\right) = 0$

$\therefore$ $\tfrac{1}{4}\omega_n\log_e 5 = n\pi$ $n = 1,2,3,\ldots$

$\therefore$ $$\lambda_n = \omega_n^2 + 5 = \frac{16n^2\pi^2}{(\log_e 5)^2} + 5$$

& $$u_n(x) = (4x+1)^{-1/2}\sin\left(\frac{n\pi\log_e(4x+1)}{\log_e 5}\right)$$

10(i) In order for ϕ to be a scalar potential for the irrotational flow of an inviscid incompressible fluid $\nabla^2\phi = 0$

i.e. $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

Now $\phi(r,\theta,z) = \left(Ar + \frac{B}{r}\right)\cos\theta + C\theta$

So $\frac{\partial\phi}{\partial r} = \left(A - \frac{B}{r^2}\right)\cos\theta$,

$\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = \left(A + \frac{B}{r^2}\right)\cos\theta$,

$\frac{\partial^2\phi}{\partial\theta^2} = -\left(Ar + \frac{B}{r}\right)\cos\theta$

& $\frac{\partial^2\phi}{\partial z^2} = 0$

$\therefore$ $$\nabla^2\phi = \frac{1}{r}\left(A + \frac{B}{r^2}\right)\cos\theta - \frac{1}{r^2}\left(Ar + \frac{B}{r}\right)\cos\theta = 0$$

(ii) $$\underset{\sim}{u} = \text{grad}\,\phi = \frac{\partial\phi}{\partial r}\underset{\sim}{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underset{\sim}{e}_\theta + \frac{\partial\phi}{\partial z}\underset{\sim}{e}_z$$

$$= \left(A - \frac{B}{r^2}\right)\cos\theta\,\underset{\sim}{e}_r - \frac{1}{r}\left(\left(Ar + \frac{B}{r}\right)\sin\theta - C\right)\underset{\sim}{e}_\theta$$

The boundary conditions are

(a) $\underset{\sim}{u}$ has no $\underset{\sim}{e}_r$ component on $r = a$,

(b) $\underset{\sim}{u} \to U\underset{\sim}{i} = U(\cos\theta\,\underset{\sim}{e}_r - \sin\theta\,\underset{\sim}{e}_\theta)$ as $r \to \infty$.

From (b)$\}$ $A = U$

From (a)$\}$ $B = Aa^2 = Ua^2$

Also circulation $\oint \underset{\sim}{u}\cdot d\underset{\sim}{r} = -\kappa$

i.e. $\oint \nabla\phi\cdot d\underset{\sim}{r} = -\kappa$

or $\phi(r,2\pi,z) - \phi(r,0,z) = -\kappa$

So $C = -\frac{\kappa}{2\pi}$

and $$\phi(r,\theta,z) = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{\kappa}{2\pi}\theta.$$