Question 8

(i) Show that the Fourier sine series for the function

$$f(x) = \cos x$$

on the interval $0 < x < \pi$ is

$$\frac{8}{\pi} \sum_{m=1}^{\infty} \frac{m \sin 2mx}{4m^2 - 1}.$$ [7]

(ii) Show that the eigenvalue problem

$$X''(x) + \lambda X(x) = 0 \qquad (0 < x < \pi),$$
$$X(0) = 0,$$
$$X(\pi) = 0$$

has eigenvalues $\lambda_n = n^2 (n = 1, 2, 3, \ldots)$ and corresponding eigenfunctions

$$X_n(x) = \sin nx.$$ [4]

(iii) The equation governing the temperature distribution $u(x, t)$ in an insulated bar of length π is given by the diffusion equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi, t > 0),$$

where x and t represent distance along the bar and time respectively and k is a positive constant. Initially the temperature distribution of the bar is

$$u(x, 0) = \cos x \qquad (0 < x < \pi).$$

At time $t = 0$ the two ends of the bar are immersed in blocks of ice and are thus maintained at 0 °C.

(a) Write down the boundary conditions at $x = 0$ and $x = \pi$ for the temperature distribution. [2]

(b) Use the method of separation of variables to determine the temperature distribution $u(x, t)$ at times $t > 0$. [7]

Question 9

Consider the regular Sturm–Liouville problem defined by the differential equation

$$\left((4x + 1)^2 u'\right)' - u + \lambda u = 0 \qquad (0 < x < 1)$$

with the boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(1) = 0.$$

(i) State the orthogonality condition which is satisfied by eigenfunctions of the problem. [1]

(ii) Find upper and lower bounds for each eigenvalue λ_n $(n = 1, 2, 3, \ldots)$ of the problem and hence show that all the eigenvalues are greater than 5. [8]

(iii) By using the change of variable

$$x = \tfrac{1}{4}(t - 1),$$

find the general solution of the differential equation in the case $\lambda > 5$ and hence solve the eigenvalue problem. [11]